CPSC 303, Homework 10 Solutions

2(a) We have $c_0 = c_n = 0$ and

$$
\begin{aligned}
2c_1 + \frac{h_1}{h_0+h_1} c_2 \qquad\qquad\qquad\qquad &= 3\Phi_1 \\
\frac{h_1}{h_1+h_2} c_1 + 2c_2 + \frac{h_1}{h_1+h_2} c_3 \qquad &= 3\Phi_2 \\
\frac{h_2}{h_2+h_3} c_2 + 2c_3 + \frac{h_3}{h_2+h_3} c_4 &= 3\Phi_3 \\
\vdots \qquad\qquad &
\end{aligned}
$$

So that

$$(2I + M)\vec{c} = 3\Phi$$

where

$$
M = \begin{bmatrix}
0 & \frac{h_1}{h_0+h_1} & & & \\
\frac{h_1}{h_1+h_2} & 0 & \frac{h_2}{h_1+h_2} & & \\
 & \frac{h_2}{h_2+h_3} & 0 & \frac{h_3}{h_2+h_3} & \\
 & & & \ddots &
\end{bmatrix}
$$

2(b) So the i^{th} row of M contains $\frac{h_{i-1}}{h_{i-1}+h_i}$, $\frac{h_i}{h_{i-1}+h_i}$ and the rest 0's

(unless $i=1$ where $\frac{h_0}{h_0+h_1}$ is not there, or $i=n-1$, where $\frac{h_{n-1}}{h_{n-2}+h_{n-1}}$ is not there).

Hence the i^{th} row has sum of absolute values at most $\frac{h_{i-1}}{h_{i-1}+h_i} + \frac{h_i}{h_{i-1}+h_i} = 1$.

Hence $\|M\|_\infty \leq 1$ (really $=1$ unless $n-1=1,2$)

So

$$(2I+M)\vec{c} = 3\Phi$$

and $\|M\|_\infty \leq 1 \Rightarrow$

$$(2I+M)^{-1} = \frac{1}{2}(I+M/2)^{-1}$$

$$= \frac{1}{2}\left(I - M/2 + (M/2)^2 - (M/2)^3 + \dots\right)$$

Since $(I+A)^{-1} = I - A + A^2 - A^3 + \dots$ whenever $\|A\|_\infty < 1$,

and $\|M/2\|_\infty \leq 1/2$.

Hence

$$\vec{c} = \frac{3}{2}\left(I - M/2 + (M/2)^2 - (M/2)^3 + \dots \right) \Phi.$$

(3) 1(b,c) of Adjacency Matrices, ...

(1b)

The idea is to run the software

sine_initially(m, ρ, T)

in the appendix for $m = 10$, $\rho = 2/3$,

and T where the software is

likely to be unstable.

Since $|1-2\rho| + 2\rho = 1/3 + 4/3 = 5/3$,

we have $(|1-2\rho| + 2\rho)^k \approx 2^{53}$ for

$k = 53(\log 2 / \log(5/3)) = 71.916\dots \approx 72$

Since $h = \frac{1}{m} = \frac{1}{10}$, we have $\rho = H/h^2$,

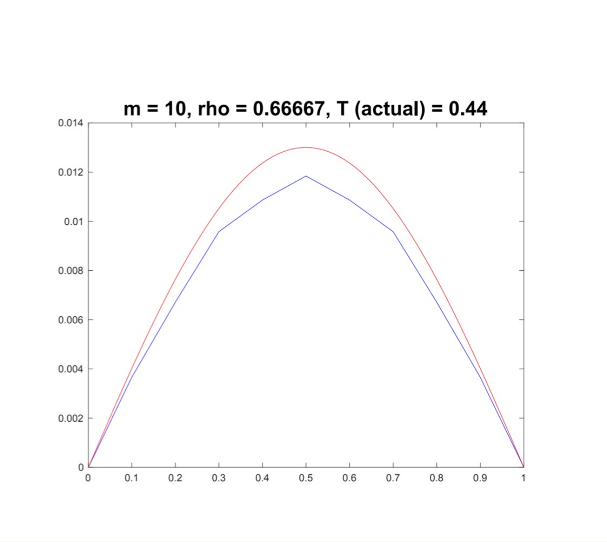
So $H = \rho h^2 = (2/3)(1/10)^2 = 2/300 = 1/150$

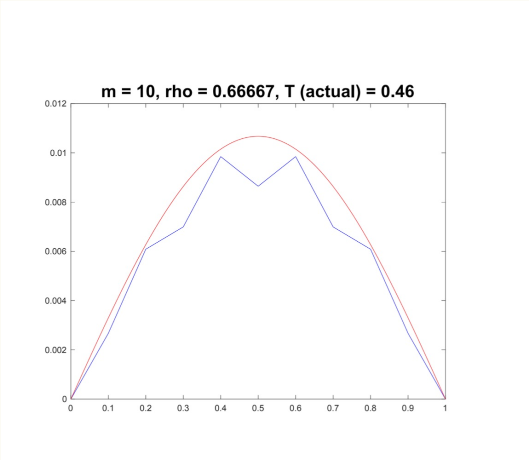
So to get 72 iterations we have

$T \simeq 72/150 = 12/25 = 0.48$

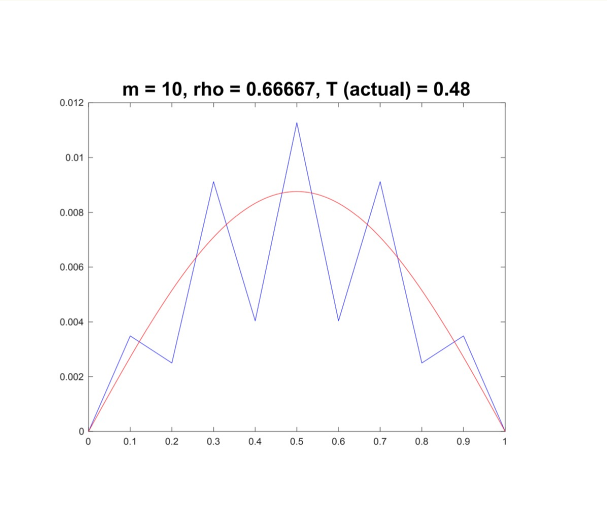
So we plot for $T = 0.44, 0.46, 0.48, 0.50$

the numerical solution up to these times, and we have:

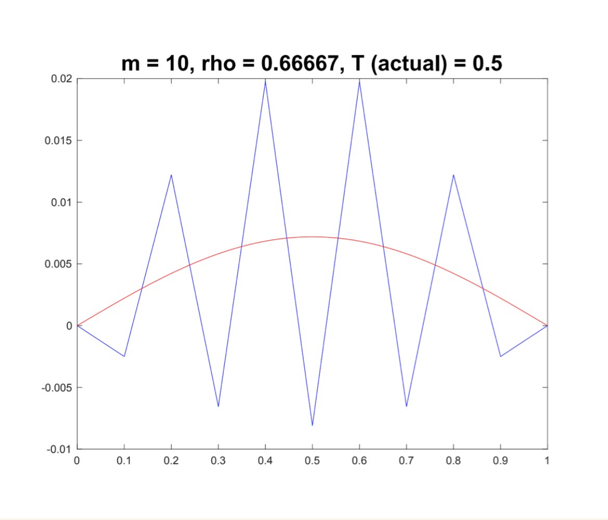




m = 10, rho = 0.66667, T (actual) = 0.46
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1



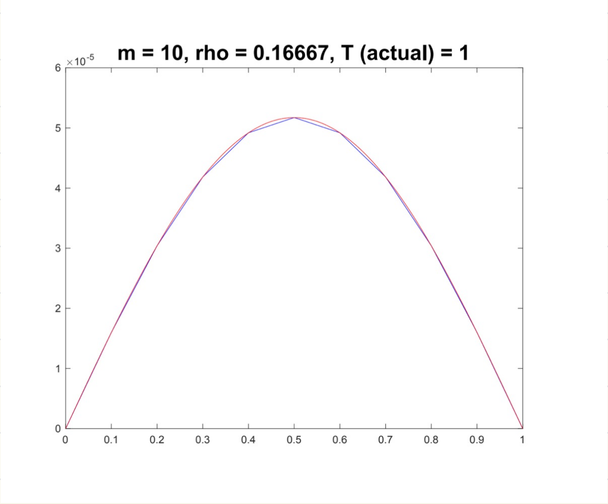
m = 10, rho = 0.66667, T (actual) = 0.48
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1



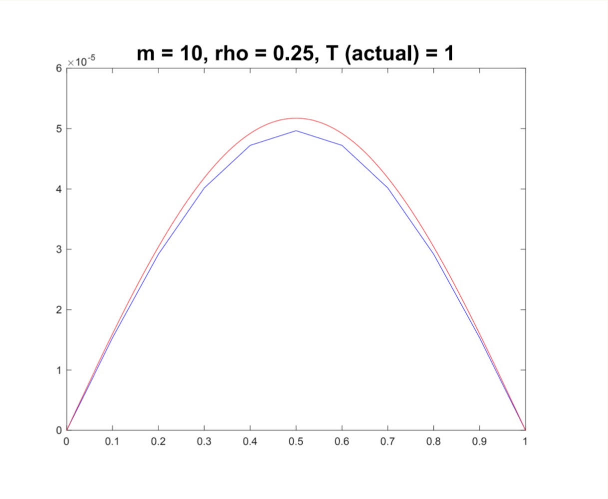
m = 10, rho = 0.66667, T (actual) = 0.5
0.02
0.015
0.01
0.005
0
-0.005
-0.01
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

(1c) Running the software where $m = 10$, ρ various, and $T = 1$ we see that the numerical solution is less than the actual if $\rho = 1/3, 1/4$, and larger if $\rho = 1/8, 1/10$, and very close to the actual if $\rho = 1/6$.

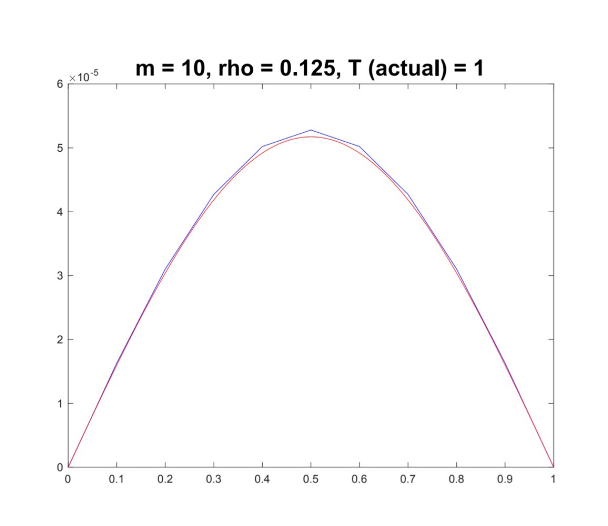
$\rho = 1/6$



$\rho = 1/4$


m = 10, rho = 0.25, T (actual) = 1
×10^-5
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$\rho = 1/8$


m = 10, rho = 0.125, T (actual) = 1
×10^-5
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

2(a) We have

$$U(i,1) =$$

$$(1-2\rho)\,U(i,0) + \rho\,\big(U(i+1,0) + U(i-1,0)\big)$$

$$= \sin(ih)\Big[(1-2\rho) + \rho\big(2\cos(\pi h)\big)\Big]$$

$$= U(i,0)\Big(1 + 2\rho\big(\cos(\pi h) - 1\big)\Big)$$

Similarly, whenever $U(i,j) = c_j \sin(ih)$ we have $U(i,j+1) =$

$$c_j \sin(ih)\,\big(1 + 2\rho(\cos(\pi h) - 1)\big).$$

Hence, by induction on j:

$$U(i,j) = U(i,0)\,\Big(1 + 2\rho\big(\cos(\pi h) - 1\big)\Big)^{j}.$$

(b, c) since $\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{4!} + O(\alpha^6)$,

for α small,

$$1 + 2\rho\left(\cos(\pi h) - 1\right)$$

$$= 1 + 2\rho\left(-\frac{(\pi h)^2}{2} + \frac{(\pi h)^4}{4!} + O(h^6)\right)$$

Since $\log(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3)$ for ε small,

$$\log\left(\left(1 + 2\rho\left(\cos(\pi h) - 1\right)\right)^{1/\rho h^2}\right)$$

$$= \frac{1}{\rho h^2}\left(\underbrace{2\rho\left(-\frac{\pi^2 h^2}{2} + \frac{\pi^4 h^4}{24} + O(h^6)\right)}\right.$$

$$\left. - \frac{1}{2}\left(\qquad\qquad\right)^2 + O(h^6)\right)$$

$$= \frac{1}{\rho h^2}\left(-\rho\pi^2 h^2 + \rho\frac{\pi^4 h^4}{12} + O(h^6)\right.$$

$$\left. - \frac{1}{2}\rho^2\pi^4 h^4 + O(h^6)\right)$$

$$= -\pi^2 + h^2 \frac{\pi^4}{12} - \frac{1}{2} \rho \pi^4 h^2 + O(h^4)$$

$$= -\pi^2 + h^2 \pi^4 \left(\frac{1}{12} - \frac{1}{2} \rho \right) + O(h^4)$$

Hence

$$\left(1 + 2\rho(\cos(\pi h) - 1) \right)^{\frac{1}{\rho h^2}} = e^{-\pi^2 + h^2 g(\rho) + O(h^4)}$$

where $g(\rho) = \pi^4 \left(\frac{1}{12} - \frac{1}{2} \rho \right)$

So $g(\rho) < 0$ for $\rho > 1/6$

$g(\rho) = 0$ " " $= 1/6$

" > 0 " " $< 1/6$

Hence the numerical method agrees with $e^{-\pi^2} + O(h^4)$ iff $\rho = 1/6$.

[This also agrees with the numerical

experiments in the previous question.]

(d) This is exactly the value of ρ making the method a higher order method in Section 4.6.