CPSC 303 Homework Solutions 1, 2024

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(2a)
$$c_0 + c_1 + c_2 = 0 \quad (A)$$
$$c_1 + 2c_2 = 1 \quad (B)$$
$$c_1 + 4c_2 = 0 \quad (C)$$

(C) implies $c_1 = -4c_2$. Then (B) implies

$$1 = c_1 + 2c_2 = -4c_2 + 2c_2 = -2c_2, \text{ so } c_2 = \frac{-1}{2}.$$

So $c_1 = -4c_2 = 2 = 4/2$.

(A) implies $c_0 = -c_1 - c_2 = -4/2 + 1/2 = -3/2$.

Alternate: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ is an example of a Vandermonde matrix, which is known to be invertible. Hence

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{pmatrix} \text{any} \\ \text{given} \\ \text{vector} \end{pmatrix}$$ has a unique solution.

Since $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -3/2 \\ 4/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, this is the unique solution.

2(b)

$$c_0 f(x_0) + c_1 f(x_0+h) + c_2 f(x_0+2h)$$

$$= (c_0+c_1+c_2)\, f(x_0)$$

$$+ (\quad c_1 + 2c_2)\, h\, f'(x_0)$$

$$+ (\quad c_1 + 4c_2)\, \frac{h^2}{4}\, f''(x_0)$$

$$+ \qquad O(h^3) M_3$$

(here the constant in the $O(h^3)$ notation really depends on c_1, c_2, so one could write $O(h^3) M_3$ as $O(h^3)(|c_1|+|c_2|) M_3$ where now $O(h^3)$ is bounded by an absolute constant).

Hence the above equals $h f'(x_0) + O(h^3) M_3$ iff c_0, c_1, c_2 satisfies the system in part (a),

i.e. $c_0 = \frac{-3}{2}$, $c_1 = \frac{4}{2}$, $c_2 = \frac{-1}{2}$.

2(c) Part (b) implies that

$$\frac{-\left(\frac{3}{2}\right) f(x_0) + \left(\frac{4}{2}\right) f(x_0+h) - \left(\frac{1}{2}\right) f(x_0+2h)}{h} = f'(x_0) + O(h^2) M_2$$

which is essentially the 3-point formula in the middle of page 411, 14.1.2 (b) in [A&G]

2(d) This would give you a formula

$$\frac{c_0 f(x_0) + c_1 f(x_0+h) + c_2 f(x_0+2h) + c_3 f(x_0+3h)}{h}$$

$= f'(x_0) + O(h^3) M_4$, i.e. a four-point

formula for $f'(x_0)$ with error term $O(h^3)$, rather than $O(h^2)$ in parts (a, b, c).

This would follow from Taylor series similar to part (b), where for

$l = 0, 1, 2, 3$ we write

$$f(x_0 + lh) = f(x_0) + \frac{lh}{1} f'(x_0) + \frac{(lh)^2}{2} f''(x_0)$$

$$+ \frac{(lh)^3}{6} f'''(x_0) + l^4 O(h^4) M_4 .$$

3(a) $y' = y^3 \Rightarrow \frac{dy}{dt} = y^3 \Rightarrow \frac{dy}{y^3} = dt$

$\Rightarrow \int \frac{dy}{y^3} = \int dt \Rightarrow \frac{y^{-2}}{-2} + C_1 = t + C_2$

or $\frac{1}{-2y^2} = t + C$ so $y^2 = \frac{1}{-2(t+C)}$

so $y(t) = y = \left(\frac{1}{-2(t+C)}\right)^{1/2}$

3(b) If $y(1) = 1$, then $\left(\frac{1}{-2(1+C)}\right)^{1/2} = y(1) = 1$

so $\frac{1}{-2(1+C)} = 1$ so $1 + C = \frac{1}{-2}$ so $C = -1 - \frac{1}{2} = \frac{-3}{2}$

So $y(t) = \left(\frac{1}{-2\left(t - \frac{3}{2}\right)}\right)^{1/2} = \left(\frac{1}{3 - 2t}\right)^{1/2}$

3(c) Similarly, $\frac{dy}{dt} = y^4 \Rightarrow \frac{dy}{y^4} = dt$

$$\Rightarrow \quad \frac{-1}{3y^3} = t + C.$$

If $y(1) = 1$, then $\frac{-1}{3} = 1 + C$ so $C = \frac{-4}{3}$

so $\frac{-1}{3y^3} = t - \frac{4}{3}$ so $y^3 = \frac{1}{-3\left(t - \frac{4}{3}\right)}$

So $y = y(t) = \left(\frac{1}{4-3t}\right)^{1/3}$

3(d) For part (b), $y(t) \to \infty$ when $t \to \frac{3}{2}$

$\left(\text{since then } 3-2t \to 0 \text{ and so } \left(\frac{1}{3-2t}\right)^{1/2} \to \infty\right)$

3(e) For part (c), $y(t) \to \infty$ when $t \to \frac{4}{3}$

(4a) A plot indicates $z(t) > y(t)$ for all $0 < t < 4/3$ (and $z(t) \to \infty$ as $t \to 4/3$).

(4b) Solution 1: Since the functions $y \mapsto y^3$ and $z \mapsto z^4$ are Lipschitz (i.e. differentiable), the equations

$$y' = y^3, \quad y(1) = 1 \quad \text{and}$$

$$z' = z^4, \quad z(1) = 1$$

can be solved via separation of variables. Hence we can use the computation in Problem 3.

4(b) Solution 2: Calculate directly:

$y(t) = (3-2t)^{-1/2}$, so $y(1) = (3-2\cdot 1)^{-1/2}$
$= 1^{-1/2} = 1$

and

$$y'(t) = \left(\frac{-1}{2}\right)(3-2t)^{-3/2}(-2)$$
$$= (3-2t)^{-3/2} = y^3,$$

and similarly for $z = z(t)$.

(c) $y'(1) = (y(1))^3 = 1^3 = 1$,

$z'(1) = (z(1))^4 = 1^4 = 1$.

(d) $y' = y^3 \implies y'' = (y^3)' = 3y^2 y'$.

Hence $y''(1) = 3(y(1))^2 y'(1) = 3$, using parts (b,c). Similarly for $z''(1)$.

(4e) Let $u(t) = z(t) - y(t)$.

Then $u(1) = z(1) - y(1) = 1 - 1 = 0$

$u'(1) = z'(1) - y'(1) = 1 - 1 = 0$

$u''(1) = z''(1) - y''(1) = 4 - 3 = 1$

Hence, by the usual calculus test,

since $u'(1) = 0$ and $u''(1) = 1$,

u has a local minimum at $t = 1$.

Since $u(1) = 0$, it follows that $u(t) > 0$

for t near 1.

4(e) Alternative (More direct)

Since $y(t)$ is continuous near $t = 1$,

and $y'(t) = (y(t))^3$, y' is continuous near $t = 1$.

Since $y'' = 3y^2 y'$, y'' " " " ".

Hence for any h we have

$$y(1+h) = y(1) + h\,y'(1) + \frac{h^2}{2} y''(\xi)$$

with ξ between 1 and $1+h$. Since

$y''(1) = 3$, and y'' is continuous,

$y''(\xi) \leq 3.01$ for ξ near 1, so

for h near 0,

$$\begin{aligned} y(1+h) &= y(1) + h\,y'(1) + \frac{h^2}{2} y''(\xi) \\ &\leq 1 + h + \frac{h^2}{2} 3.01 . \end{aligned}$$

Since $z''(1) = 4$ (and $z(1) = z'(1) = 1$)

we similarly have that for h near 0,

$$\begin{aligned} z(1+h) &\geq 1 + h + \frac{h^2}{2} 3.99 \\ &\geq 1 + h + \frac{h^2}{2} 3.01 \geq y(1+h). \end{aligned}$$

Note: This holds for $|h|$ near 0, so

h can be negative.

Remark: Since $y(1) = y'(1) = 1$ and $y''(1) = 3$, we really know that

$$y(1+h) = y(1) + h\,y'(1) + \frac{h^2}{2}\left(y''(1) + o(h)\right)$$

where $o(h)$ is a function $\to 0$ as $h \to 0$.

This is the "little-oh" notation $o(h)$, which is convenient.