Practice Homework 5, 2024

(1) $y_{n+2} - 7y_{n+1} + 12y_n = 0$ (homogeneous form)

is $(r^2 - 7r + 12)(y_n) = 0$

$$r^2 - 7r + 12 = (r-3)(r-4)$$

So $y_n = c_1 3^n + c_2 4^n$

"Guess" $x_n = an + b$ as a solution

to $x_{n+2} - 7x_{n+1} + 12x_n = n + 5$

so

$$\big(a(n+2)+b\big) - 7\big(a(n+1)+b\big) + 12(an+b) = n+5$$

so

$$n(6a) + (6b - 5a) = n + 5$$

$$a = \frac{1}{6}, \quad b = \frac{5+5a}{6} = \frac{35/6}{6} = \frac{35}{36}$$

So $x_n = \frac{1}{6}n + \frac{35}{36}$ is one solution.

Hence general solution is

$$x_n = \frac{1}{6}n + \frac{35}{36} + c_1 3^n + c_2 4^n$$

(2) Homogeneous ODE is

$$\left(\left(\frac{d}{dt}\right)^2 - 7\left(\frac{d}{dt}\right) + 12 \right) z = 0$$

so

$$\left(\frac{d}{dt} - 3\right)\left(\frac{d}{dt} - 4\right) z = 0$$

so

$$z = c_1 e^{3t} + c_2 e^{4t}$$

"Guess" $y(t) = at + b$ for particular solution, so

$$y''(t) - 7y'(t) + 12y(t) = t + 3$$

$$0 - 7a + 12(at+b) = t + 3$$

$$t(12a) + 12b - 7a = t + 3$$

So $12a = 1$, $a = \frac{1}{12}$

$$12b = 7a + 3 = \frac{7}{12} + 3 = \frac{43}{12}$$

So $$b = \frac{43}{144}$$

So $$y(t) = \frac{1}{12}t + \frac{43}{144}$$ is

a particular solution. So general solution is

$$y(t) = \frac{1}{12}t + \frac{43}{144} + c_1 e^{3t} + c_2 e^{4t}$$

(3) Solution to

$$y_{n+2} - 6y_{n+1} + 9y_n = 0$$

solve

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

So general solution is

$$y_n = c_1 3^n + c_2 n 3^n$$

"Guess" $x_n = an+b$ for a solution to

$$x_{n+2} - 6x_{n+1} + 9x_n = n+5$$

$$a(n+2)+b - 6(a(n+1)+b) + 9(an) = n+5$$

$$n(4a) + (4b-4a) = n+5$$

$$a = \frac{1}{4}, \quad b = \frac{5-4a}{4} = \frac{4}{4} = 1$$

So

$$x_n = \frac{1}{4}n + 1 \quad \text{works.}$$

Hence general solution is

$$x_n = \frac{1}{4}n + 1 + c_1 3^n + c_2 n 3^n$$

(4) Homogeneous ODE is

$$\left(\frac{d}{dt} - 3\right)^2 z = 0$$

so $\quad z = c_1 e^{3t} + c_2 t e^{3t}$

Guess $y = at + b$ for particular solution!

$$y'' - 6y' + 9y = t + 3$$

$$0 - 6a + 9(at + b) = t + 3$$

$$(9a)t + (9b - 6a) = t + 3$$

So $\quad a = \frac{1}{9}, \quad b = \frac{3 + 6a}{9} = \frac{3 + 2/3}{9} = \frac{11}{27}$

so

$$y(t) = \frac{1}{9}t + \frac{11}{27} \quad \text{works}$$

So general solution is

$$y(t) = \frac{1}{9}t + \frac{11}{27} + c_1 e^{3t} + c_2 t e^{3t}$$

(5) General solution given by

solution to $(\sigma - 4)(x_n) = 0$ and

solution to $(\sigma - 5)^2 (x_n) = 0$

so
$$x_n = c_1 4^n + c_2 5^n + c_3 n 5^n.$$

Given x_0, x_1, x_2 we have

$$c_1 4^0 + c_2 5^0 + c_3 0 \cdot 5^0 = x_0$$
$$c_1 4^1 + c_2 5^1 + c_3 1 \cdot 5^1 = x_1$$
$$c_1 4^2 + c_2 5^2 + c_3 2 \cdot 5^2 = x_2$$

So
$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 4 & 5 & 5 \\ 16 & 25 & 50 \end{bmatrix}}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

This matrix is invertible, since

$$\det = 1\cdot 5\cdot 50 + 1\cdot 5\cdot 16 - 1\cdot 5\cdot 25 - 1\cdot 4\cdot 50$$

$$= 250 + 80 - 125 - 200$$

$$= 330 \quad - 325 \quad \neq 0$$

(6)

$$\left(\frac{d}{dt} - 3\right) p(t)e^{3t}$$

$$= \frac{d}{dt}\left(p(t)e^{3t}\right) - 3p(t)e^{3t}$$

$$= \left(p'(t)e^{3t} - p(t)e^{3t}3\right) - 3p(t)e^{3t}$$

$$= p'(t)e^{3t}$$

$$= dc_d t^{d-1} + c_{d-1}(d-1)t^{d-2} + \cdots$$

is of degree $d-1$.

(7) (a)

$$(1/2)^{1074} = 0.\underbrace{00\ldots0}_{51 \text{ zeros}}1 \times 10^{-1022}$$

So

$$(3/2)(1/2)^{1074}$$

$$= \underbrace{(1.1)}_{\text{in base } 2}(1/2)^{1074} = 0.\underbrace{0\ldots}_{51 \text{ zeros}}011 \times 10^{-1022}$$

But double precision only allows 52 bits past $0.b_1 b_2 \ldots b_{52} \times 10^{-1022}$, not 53 bits. Hence $(3/2) \times (1/2)^{1074}$ can be stored as is in double precision.

(b) Yes, since

$$(3/2) \times (1/2)^{1073}$$

$$= 0.\underbrace{0\ldots0}_{\text{50 bits}}\underbrace{11}_{} \times 10^{-1022}$$

51st and 52nd bit.

(c) 2^{1073} is reported as Inf (Infinity)

so $(3/2)\,(1/2)^{1073}\,2^{1073}$ evaluates to

$(3/2)\cdot(1/2)^{1073}\cdot \text{Inf}$, evaluates to

1. $\Big(\underbrace{(3/2)\,(1/2)^{1073}}_{\text{non-zero}}\Big)\,\text{Inf} = \text{Inf}$

OR

2. $(3/2)\,\Big(\underbrace{(1/2)^{1073}\,\text{Inf}}\Big)$

$(3/2)\quad \text{Inf} \quad = \text{Inf}$

So either way you get Inf.

(d) Since $3/2$, $(1/2)^{1073}$, and $(3/2)(1/2)^{1073}$

can be recorded exactly, as can

$(2)^{1000} \quad (= 1.00\ldots \times 2^{1000})$

$\left(\left(3/2\right)\left(1/2\right)^{1073}\, 2^{1000}\right)$ will evaluate

(either way the multiplication is done)

to $\left(3/2\right)\left(1/2\right)^{73}$ (exactly).

Hence this times 2^{73} evaluates to

$3/2$ exactly $\quad (= 1.1 \times 2^{0}$ in binary$)$

(e) $5/4 = 1.01$, so

$$\left(5/4\right) \times \left(1/2\right)^{1073} = 0.\overbrace{00\ldots0}^{50 \text{ bits}}101 \times 2^{-1022}$$

51^{st} bit

52^{nd} bit

53^{rd} bit

Since double precision can't remember the 53^{rd} bit, which is non-zero, double precision can't store this number exactly.