# GROUP HOMEWORK 10, CPSC 303, SPRING 2024

JOEL FRIEDMAN



# This homework will not be collected or graded

---

(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Consider a natural cubic spline to approximate $f(x)$ at the values $x = x_0, x_1, \ldots, x_n$. When $h_1 = h_2 = \cdots = h_{n-1}$, we wrote this system of equatios as

$$(2I + N_{\mathrm{rod},n-1}/2)\mathbf{c} = 3\Phi,$$

where $\Phi$ was a vector of divided differences (see March 22, or see the top of page 343 of [A&G], dividing the $i$-th row of this system by $h_{i-1} + h_i$).

(a) If $h_1, \ldots, h_{n-1}$ are not necessarily equal, write this system as $(2I + M)\mathbf{c} = 3\Phi$, for some matrix $M$.

(b) If $h_1, \ldots, h_{n-1}$ are not necessarily equal, and $M$ is as in part (a), can we still assert that

$$\mathbf{c} = (3/2)\bigl(I - (M/2) + (M/2)^2 - (M/2)^3 + \cdots\bigr)\Phi \ ?$$

Explain.

(3) Problems 1(b,c) on CPSC 303: Adjacency Matrices, Splines, and the Heat Equation.

(4) Problem 2 on CPSC 303: Adjacency Matrices, Splines, and the Heat Equation.

---

Research supported in part by an NSERC grant.

(5) **Bonus Question, worth 0.1 point, may be submitted any time before the last day of classes.** If $f, g \in C[0,1]$ (i.e., both are continuous functions $[0,1] \to \mathbb{R}$, we define their "dot product"[1] to be[2]

$$\langle f, g\rangle = \int_0^1 f(x)g(x), \mathrm{d}x.$$

We say that the sequence $f_1, f_2, \ldots \in C[A,B]$ *converges weakly* to a function $f \in C[A,B]$ if for any $g \in C[A,B]$ we have

$$\langle f_n, g\rangle \xrightarrow{n\to\infty} \langle f, g\rangle$$

(a) Show that if $g$ is any *piecewise constant* function on $[0,1]$ ([A&G], page 355), we have

$$\int_0^1 \sin(nx)g(x) \xrightarrow{n\to\infty} 0.$$

[Hint: First check this when for some $a, b \in [0,1]$ we have $g(x)$ is 1 for $a \le x < b$ and 0 otherwise.[3] Then use the linearity in $g$ of $\langle f, g\rangle$.]

(b) Show that if $g_1, g_2 \in C[0,1]$ and

$$\epsilon = \max_{0\le x\le 1} |g_1(x) - g_2(x)|,$$

then

$$-\epsilon \le \langle g_1 - g_2, \sin(nx)\rangle \le \epsilon.$$

(c) Show that $f_n = \sin(nx)$ converges weakly to 0. (This requires advanced calculus[4].)

(d) Show that $f_1, f_2, \ldots$ is any sequence in $C[0,1]$ such that $\langle f_i, f_i\rangle \le 1$, then there is a subsequence of $f_1, f_2, \ldots$ that weakly converges to a function in $L^2[0,1]$, where weak convergence in $L^2[0,1]$ is defined analogously. (This requires some modern analysis[5] (also known as real analysis), if only to understand what is meant by $L^2[0,1]$ and basic properties of Hilbert spaces.) [Hint: Use the fact that $L^2[0,1]$ is a separable Hilbert space.] [Remark: Because there is only one infinite dimensional, separable Hilbert space (up to isomorphism), you are proving that the unit ball in any infinite dimensional, separable Hilbert space is weakly compact. Of course, the unit ball in any finite dimensional Hilbert space is compact.]

---

[1] This is an analog of the dot product $\mathbf{s}\cdot\mathbf{t} = s_1t_1 + \cdots + s_nt_n$ of vectors in $\mathbb{R}^n$. Such analogs are often called "inner products."

[2] Hence

$$\|f\|_2 = \sqrt{\langle f, f\rangle}$$

in the notation of [A&G], Section 12.1, page 366. Much of the rest of Chapter 12 of [A&G] can be written in terms of this "dot product," or the weighted dot product $\langle f, g\rangle_w \stackrel{\mathrm{def}}{=} \int_0^1 f(x)g(x)w(x)\,dx$ for an everywhere positive function "weight function" $w \in C[0,1]$. The only difference is that you might have to replace $C[0,1]$ with $C[a,b]$ for real $a < b$.

[3] Here we are using the particular definition of "piecewise constant" in [A&G]; the same is true for any variant of this definition.

[4] See, for example, *Advanced Calculus* by Avner Friedman.

[5] See, for example, *Foundations of Modern Analysis* by Avner Friedman.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA.
*E-mail address*: jf@cs.ubc.ca
*URL*: http://www.cs.ubc.ca/~jf