GROUP HOMEWORK 9, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Consider the natural cubic spline, $v(x)$, through the 3 points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$. Show that $v(x)$ is three times differentiable iff $f[x_0, x_1, x_2] = 0$. [Hint: in the notation in class and [A&G], bottom page 340 to middle page 342 (Section 11.3), $v(x)$ is composed of $s_0(x)$ and $s_1(x)$. Since $s_0'''(x) = d_0$ and $s_1'''(x) = d_1$, we need to see when $d_0 = d_1$. Use the formulas in the textbook or class on March 22.]

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(3) For any $n \in \mathbb{N} = \{1, 2, \ldots\}$, we define $N_{\mathrm{rod},n}$ as usual, namely

$$N_{\mathrm{rod},n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In class we defined the (simple, undirected) graph called the "path of length n," denoted P_n (which has vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$). Use the fact that $N_{\mathrm{rod},n}$ is the adjacency matrix, A_{P_n}, of P_n, to compute:

(a) the i-th row of $N^4_{\mathrm{rod},n}$ when $5 \le i \le n-4$, i.e., describe the (i,j)-th entry of $N^4_{\mathrm{rod},n}$ for j between $i-4$ and $i+4$ (i.e., do this by counting the number of walks of length 4 from i to j); and

(b) the first row of $N^4_{\mathrm{rod},n}$, assuming $n \ge 5$ (i.e., do this by counting the number of walks of length 4 from 1 to $j = 1, 2, 3, 4, 5$).

[Part (a) above is vacuous unless $n \ge 9$.]

(4) There is no question 4 on this homework.

(5) **Bonus Question, worth 0.1 point, may be submitted any time before the last day of classes.** If $f, g \in C[0,1]$ (i.e., both are continuous functions $[0,1] \to \mathbb{R}$, we define their "dot product"[1] to be[2]

$$\langle f, g\rangle = \int_0^1 f(x)g(x), \mathrm{d}x.$$

We say that the sequence $f_1, f_2, \ldots \in C[A,B]$ *converges weakly* to a function $f \in C[A,B]$ if for any $g \in C[A,B]$ we have

$$\langle f_n, g\rangle \xrightarrow{n\to\infty} \langle f, g\rangle$$

(a) Show that if g is any *piecewise constant* function on $[0,1]$ ([A&G], page 355), we have

$$\int_0^1 \sin(nx)g(x) \xrightarrow{n\to\infty} 0.$$

[1]This is an analog of the dot product $\mathbf{s} \cdot \mathbf{t} = s_1t_1 + \cdots + s_nt_n$ of vectors in $\mathbb{R}^n$. Such analogs are often called "inner products."

[2] Hence

$$\|f\|_2 = \sqrt{\langle f, f\rangle}$$

in the notation of [A&G], Section 12.1, page 366. Much of the rest of Chapter 12 of [A&G] can be written in terms of this "dot product," or the weighted dot product $\langle f, g\rangle_w \stackrel{\mathrm{def}}{=} \int_0^1 f(x)g(x)w(x)\,dx$ for an everywhere positive function "weight function" $w \in C[0,1]$. The only difference is that you might have to replace $C[0,1]$ with $C[a,b]$ for real $a < b$.

[Hint: First check this when for some $a, b \in [0,1]$ we have $g(x)$ is 1 for $a \leq x < b$ and 0 otherwise.[3] Then use the linearity in g of $\langle f, g\rangle$.]

(b) Show that if $g_1, g_2 \in C[0,1]$ and
$$\epsilon = \max_{0\leq x\leq 1} |g_1(x) - g_2(x)|,$$
then
$$-\epsilon \leq \langle g_1 - g_2, \sin(nx)\rangle \leq \epsilon.$$

(c) Show that $f_n = \sin(nx)$ converges weakly to 0. (This requires advanced calculus[4].)

(d) Show that $f_1, f_2, \ldots$ is any sequence in $C[0,1]$ such that $\langle f_i, f_i\rangle \leq 1$, then there is a subsequence of $f_1, f_2, \ldots$ that weakly converges to a function in $L^2[0,1]$, where weak convergence in $L^2[0,1]$ is defined analogously. (This requires some modern analysis[5] (also known as real analysis), if only to understand what is meant by $L^2[0,1]$ and basic properties of Hilbert spaces.) [Hint: Use the fact that $L^2[0,1]$ is a separable Hilbert space.] [Remark: Because there is only one infinite dimensional, separable Hilbert space (up to isomorphism), you are proving that the unit ball in any infinite dimensional, separable Hilbert space is weakly compact. Of course, the unit ball in any finite dimensional Hilbert space is compact.]

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[3] Here we are using the particular definition of "piecewise constant" in [A&G]; the same is true for any variant of this definition.

[4] See, for example, *Advanced Calculus* by Avner Friedman.

[5] See, for example, *Foundations of Modern Analysis* by Avner Friedman.