GROUP HOMEWORK 8, CPSC 303, SPRING 2024

JOEL FRIEDMAN

Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

For this problem set, "the handout" refers to the article "CPSC 303: What the Condition Number Does and Does Not Tell Us."

(1) (0 to -8 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Let $p(x) = c_0 + c_1 x + c_2 x^2$ be a polynomial of degree at most 2, and $x_0 < x_1 < x_2$ be real numbers.
   (a) Compute the divided difference $p[x_2, x_1] - p[x_1, x_0]$.
   (b) Write c_2 as a function of $p[x_2, x_1] - p[x_1, x_0]$ and of x_0, x_1, x_2. What is this function?

   [The point of the above exercise is to suggest how Newton may have come up with the formula for interpolation with divided differences.]

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(3) Let $A < B$ be real numbers, and let $w \in C^2[A,B]$ be a twice differentiable function, and let

$$\text{Energy}_{2,w}(u) = \int_A^B w(x)\bigl(u''(x)\bigr)^2\,dx\ .$$

Hence if $w(x) = 1$ on all of $[A,B]$,

$$\text{Energy}_{2,f}(u) = \int_A^B \bigl(u''(x)\bigr)^2\,dx$$

is the usual $\text{Energy}_2(u)$ used to describe cubic splines. (In this context, w is call the "weight" or "weighting" of the above "Energy.")

(a) Show that if $v, g \in C^2[A,B]$, and $\epsilon \in$ (which intuitively we think of as small)

$$\text{Energy}_{2,w}(v + \epsilon g) = \alpha + \epsilon\beta + \epsilon^2\gamma$$

where

$$\alpha = \text{Energy}_{2,w}(v), \quad \gamma = \text{Energy}_{2,w}(g),$$

and

$$\beta = 2\int_A^B \bigl(w(x)v''(x)g''(x)\bigr)\,dx.$$

(b) Show that if for all $\epsilon \in \mathbb{R}$ we have

$$\text{Energy}_{2,w}(v) \le \text{Energy}_{2,w}(v + \epsilon g)$$

then

$$\int_A^B \bigl(w(x)v''(x)g''(x)\bigr)\,dx = 0.$$

(c) Show that if, in addition $w, v \in C^4[A,B]$ and

$$g(A) = g'(A) = g(B) = g'(B) = 0,$$

then

$$\int_A^B \bigl(w(x)v''(x)\bigr)''g(x) = 0.$$

[Hint: mimic the argument for Energy_1 done in class on March 18, i.e., use integration by parts.]

(d) Let $A = x_0 < x_1 < \cdots < x_n = B$ be real numbers, and $y_0, \ldots, y_n \in \mathbb{R}$, and

$$\mathcal{U}_{\mathbf{x},\mathbf{y}} = \bigl\{u \in C^2[A,B]\ \bigm|\ u(x_0) = y_0, \ldots, u(x_n) = y_n\bigr\}.$$

Explain that if $\text{Energy}_{2,w}\colon \mathcal{U}_{\mathbf{x},\mathbf{y}} \to \mathbb{R}$ is minimized at $v \in \mathcal{U}_{\mathbf{x},\mathbf{y}}$, and if $v \in C^4[A,B]$ and $\bigl(w(x)v''(x)\bigr)''$ is a continuous function, then for any x with $x_0 < x < x_1$ we have

$$\bigl(w(x)v''(x)\bigr)'' = 0.$$

[Hint: you can assume that for any reals $a < b$ there is a function g that is infinity differentiable, $g = 0$ for $x \notin [a,b]$, and $g \ge 0$ for $x \in [a,b]$, and g is not everywhere 0.]

(e) Explain why in case $w(x) = 1$ everywhere, v is a cubic polynomial from x_0 to x_1.

(f) Explain why for general $w(x)$ such that is never 0 for $x_0 \le x \le x_1$, we must have

$$v''(x) = \frac{c_0 + c_1 x}{w(x)}$$

for all $x_0 \le x \le x_1$.

(4) For reasons that will become clear later, for any $n \in \mathbb{N} = \{1, 2, \ldots\}$ we define $N_{\mathrm{rod},n}$ to be the matrix $\mathbb{R}^{n\times n}$, i.e., the $n \times n$ square matrix

$$N_{\mathrm{rod},n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

For example, we have

$$N_{\mathrm{rod},2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad N_{\mathrm{rod},3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad N_{\mathrm{rod},4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

One can alternatively describe $N_{\mathrm{rod},n}$ by its entries: for all $i, j \in [n]$, the (i, j)-th component of

$$\left(N_{\mathrm{rod},n}\right)_{i,j} = \begin{cases} 1 & \text{if } i - j = \pm 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $\|N_{\mathrm{rod},n}\|_\infty$?

(b) In class we will explain that if A is any $n \times n$ matrix with $\|A\|_\infty < 1$ and I_n is the $n \times n$ identity matrix, then then $(I_n - A)^{-1}$ exists, and

$$(I_n - A)^{-1} = I_n + A + A^2 + \ldots$$

(and the right-hand-side is a convergent infinite sum of matrices). Using this fact, show that for any n we have that

$$\left(4I_n + N_{\mathrm{rod},n}\right)^{-1}$$

exists and equals

$$\left(4I_n + N_{\mathrm{rod},n}\right)^{-1} = (1/4)\big(I_n - (N_{\mathrm{rod},n}/4) + (N_{\mathrm{rod},n}/4)^2 - (N_{\mathrm{rod},n}/4)/4)^3 + \cdots\big)$$

[Hint: What is $\left(I_n + N_{\mathrm{rod},n}/4\right)^{-1}$?]

(c) Use MATLAB to compute $N^2_{\mathrm{rod},n}$ for $n = 3, 4, 5$ (or you may do this by hand, without MATLAB). What appears to be a general formula for $N^2_{\mathrm{rod},n}$? (You don't have to prove it). [Hint: you might find the following MATLAB phrases useful: `M=zeros(n)` and
`for i=1:n-1 ; M(i,i+1)=1; M(i+1,i)=1; end`

(d) Similarly compute $N^3_{\mathrm{rod},n}$ for small n to give what appears to be a general formula for $N^3_{\mathrm{rod},n}$.

(5) **Bonus Question, worth 0.1 point, may be submitted any time before the last day of classes.** If $f, g \in C[0,1]$ (i.e., both are continuous functions $[0,1] \to \mathbb{R}$, we define their "dot product"[1] to be[2]

$$\langle f, g\rangle = \int_0^1 f(x)g(x), \mathrm{d}x.$$

We say that the sequence $f_1, f_2, \ldots \in C[A,B]$ *converges weakly* to a function $f \in C[A,B]$ if for any $g \in C[A,B]$ we have

$$\langle f_n, g\rangle \xrightarrow{n\to\infty} \langle f, g\rangle$$

(a) Show that if g is any *piecewise constant* function on $[0,1]$ ([A&G], page 355), we have

$$\int_0^1 \sin(nx)g(x) \xrightarrow{n\to\infty} 0.$$

[Hint: First check this when for some $a, b \in [0,1]$ we have $g(x)$ is 1 for $a \le x < b$ and 0 otherwise.[3] Then use the linearity in g of $\langle f, g\rangle$.]

(b) Show that if $g_1, g_2 \in C[0,1]$ and

$$\epsilon = \max_{0\le x\le 1} |g_1(x) - g_2(x)|,$$

then

$$-\epsilon \le \langle g_1 - g_2, \sin(nx)\rangle \le \epsilon.$$

(c) Show that $f_n = \sin(nx)$ converges weakly to 0. (This requires advanced calculus[4].)

(d) Show that $f_1, f_2, \ldots$ is any sequence in $C[0,1]$ such that $\langle f_i, f_i\rangle \le 1$, then there is a subsequence of $f_1, f_2, \ldots$ that weakly converges to a function in $L^2[0,1]$, where weak convergence in $L^2[0,1]$ is defined analogously. (This requires some modern analysis[5] (also known as real analysis), if only to understand what is meant by $L^2[0,1]$ and basic properties of Hilbert spaces.) [Hint: Use the fact that $L^2[0,1]$ is a separable Hilbert space.] [Remark: Because there is only one infinite dimensional, separable Hilbert space (up to isomorphism), you are proving that the unit ball in any infinite dimensional, separable Hilbert space is weakly compact. Of course, the unit ball in any finite dimensional Hilbert space is compact.]

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[1] This is an analog of the dot product $\mathbf{s}\cdot\mathbf{t} = s_1t_1 + \cdots + s_nt_n$ of vectors in $\mathbb{R}^n$. Such analogs are often called "inner products."

[2] Hence

$$\|f\|_2 = \sqrt{\langle f, f\rangle}$$

in the notation of [A&G], Section 12.1, page 366. Much of the rest of Chapter 12 of [A&G] can be written in terms of this "dot product," or the weighted dot product $\langle f, g\rangle_w \stackrel{\text{def}}{=} \int_0^1 f(x)g(x)w(x)\,dx$ for an everywhere positive function "weight function" $w \in C[0,1]$. The only difference is that you might have to replace $C[0,1]$ with $C[a,b]$ for real $a < b$.

[3] Here we are using the particular definition of "piecewise constant" in [A&G]; the same is true for any variant of this definition.

[4] See, for example, *Advanced Calculus* by Avner Friedman.

[5] See, for example, *Foundations of Modern Analysis* by Avner Friedman.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA.
E-mail address: jf@cs.ubc.ca
URL: http://www.cs.ubc.ca/~jf