GROUP HOMEWORK 6, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

For this problem set, "the handout" refers to the article "CPSC 303: What the Condition Number Does and Does Not Tell Us."

(1) (0 to -8 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Consider a monomial interpolation $p(x) = c_0 + c_1 x + c_2 x^2$ to data points (x_0, y_0), (x_1, y_1), (x_2, y_2), where $x_0 = 2$, $x_1 = 2 + \epsilon$, and $x_2 = 2 - \epsilon$ (but y_0, y_1, y_2 are arbitrary). Hence we are solving the equations:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2+\epsilon & 4+4\epsilon+\epsilon^2 \\ 1 & 2-\epsilon & 4-4\epsilon+\epsilon^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

(a) Show that c_2, in terms of y_0, y_1, y_2, is given by

$$c_2(\epsilon) = \frac{y_1 + y_2 - 2y_0}{2\epsilon^2}. \tag{1}$$

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(b) Now assume that for some twice differentiable f we have $y_i = f(x_i)$, hence
$$y_0 = f(2),\ y_1 = f(2+\epsilon),\ y_2 = f(2-\epsilon).$$
Use L'Hôpital's Rule or Taylor's Theorem to show that
$$\lim_{\epsilon\to 0} c_2(\epsilon) = f''(2)/2$$
(if you use Taylor's Theorem, for simplicity assume that f''' exists and is bounded near 2). [Hint: See Section 1.4 of the handout for a similar example.]

(c) How is your formula for c_2 related to the *centered formula for the second derivative*, page 412, Subsection 14.1.4, of [A&G]?

(3) Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
The point of this exercise is to carefully prove that
$$\|A\|_\infty = \max\bigl(|a|+|b|, |c|+|d|\bigr).$$
Notice that for any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ we have
$$A\mathbf{x} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

(a) Show that if $m = \|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|)$, then
$$|ax_1 + bx_2| \le m(|a|+|b|).$$

(b) Using part (a), and the same with a, b replaced with c, d, show that
$$\|A\mathbf{x}\|_\infty \le \max\bigl(|a|+|b|, |c|+|d|\bigr)\|\mathbf{x}\|_\infty.$$

(c) Show that there is an $\mathbf{x}$ with $\|\mathbf{x}\|_\infty = 1$ such that
$$\|A\mathbf{x}\|_\infty \ge |a|+|b|.$$
[Hint: take $\mathbf{x} = (\pm 1, \pm 1)$ with appropriately chosen signs.]

(d) Conclude from all the above (and perhaps replacing a, b with c, d somewhere) that
$$\|A\|_\infty = \max\bigl(|a|+|b|, |c|+|d|\bigr).$$

(4) Let $1 \le p, q \le \infty$ satisfy $(1/p) + (1/q) = 1$ (so $(p,q) = (2,2)$ is one possibility, but we also allow (p,q) equal to $(1,\infty)$ and $(\infty,1)$). Then it is known that for any $m \times n$ matrix A (with real entries) we have
$$\|A^{\mathrm{T}}\|_p = \|A\|_q, \tag{2}$$
where A^{T} is the transpose of A. Using this fact, and the previous exercise, for
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
find a formula for $\|A\|_1$. [Hint: it should match the formula in Section 6 of the handout.] [This formula for $\|A\|_1$ is not too hard to prove from scratch, but it is probably easier to use the previous exercise and (2).]

(5) There is a standard formula[1] for the determinant of a Vandermonde matrix: namely, if

$$X = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix},$$

then

$$\det(X) = \prod_{0 \le i < j \le n} (x_j - x_i).$$

This implies that if $x_0, \ldots, x_n$ are distinct, then X is invertible, and a standard formula for X^{-1} then implies[2] that and the bottom right entry of X^{-1} is

$$\left(X^{-1}\right)_{n+1,n+1} = \prod_{0 \le i \le n-1} \frac{1}{x_n - x_i} \tag{3}$$

(and similarly, up to $\pm$, for all entries of the bottom row of X^{-1}). Consider the special case

$$A(\epsilon) = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2+\epsilon & 4+4\epsilon+\epsilon^2 \\ 1 & 2-\epsilon & 4-4\epsilon+\epsilon^2 \end{bmatrix}$$

(a) Use (3) to show that for all $|\epsilon| < 1$,

$$\|A^{-1}(\epsilon)\|_\infty \ge 1/(2\epsilon^2);$$

you may use the analog of Problem 3 for 3×3 matrices and/or the fact from class and the handout (page 12) that if M is the maximum absolute value of an entry of an $n \times n$ matrix B, then $M \le \|B\|_p \le nM$ for any $1 \le p \le \infty$.

(b) Use the formula (1) to determine the entire bottom row of $A^{-1}(\epsilon)$, and hence double check the formula (3) in this case.

(c) Show that for some constant, $C > 0$, we have for all $|\epsilon| < 1$,

$$\|A(\epsilon)\|_\infty \, \|A^{-1}(\epsilon)\|_\infty \ge C/\epsilon^2$$

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[1] This formula is not hard to prove by induction on n, if you note that replacing x_n by a variable x, then the above determinant is a degree n polynomial in x with roots $x_0, \ldots, x_{n-1}$.

[2] The formula is $X^{-1} = \det(X)\text{adjugate}(X)$, where the adjugate matrix is formed by X's *cofactors*, i.e., determinants of submatrices of X where a single row and a single column are discarded.