GROUP HOMEWORK 5, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) The point of this exercise is to define a type of ODE known as *central force problem*, and to show that such ODE's satisfy a *conservation of energy.* In this exercise $\| \ \|$ refers to the L^2 norm $\| \ \|_2$. Let $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^2$, i.e., $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$, where x_1, x_2 are functions $\mathbb{R} \to \mathbb{R}$, and similary for $\mathbf{z} = \mathbf{z}(t) = (z_1(t), z_2(t))$.

(a) With the usual dot product:
$$\mathbf{x} \cdot \mathbf{z} = x_1 z_1 + x_2 z_2$$
(hence all the above depend on t), show that
$$\frac{d}{dt}(\mathbf{x} \cdot \mathbf{z}) = \mathbf{x} \cdot \dot{\mathbf{z}} + \dot{\mathbf{x}} \cdot \mathbf{z},$$
where $\dot{}$ denotes d/dt (as usual in celestial mechanics).

(b) Show that
$$\frac{d}{dt}\bigl(\|\mathbf{x}\|^2\bigr) = \frac{d}{dt}\bigl(\mathbf{x} \cdot \mathbf{x}\bigr) = 2\,\mathbf{x} \cdot \dot{\mathbf{x}}.$$

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(c) Show that if m is a constant, then

$$\frac{d}{dt}\bigl(m\|\dot{\mathbf{x}}\|^2\bigr) = 2m\,\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}.$$

(d) Show that

$$\frac{d}{dt}\|\mathbf{x}\| = \frac{d}{dt}\sqrt{x_1^2+x_2^2} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}\cdot\dot{\mathbf{x}}.$$

(e) Show that if $U\colon(0,\infty)\to\mathbb{R}$ is a differentiable function, whose derivative is u, then

$$\frac{d}{dt}U\bigl(\|\mathbf{z}\|\bigr) = u\bigl(\|\mathbf{z}\|\bigr)\frac{1}{\|\mathbf{z}\|}\,\mathbf{z}\cdot\dot{\mathbf{z}}.$$

(f) We say that a function $\mathbf{x}\colon\mathbb{R}\to\mathbb{R}^2$ satisfies a *central force law* if for some real $m>0$ and $u\colon(0,\infty)\to\mathbb{R}$ we have

$$m\ddot{\mathbf{x}} = -m\,u\bigl(\|\mathbf{x}\|\bigr)\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{1}$$

(at times u may extend to a function $[0,\infty)\to\mathbb{R}$, but for Newton's Law of Gravitation, $u(0)=+\infty$). Show that in this case

$$\text{Energy} = \text{Energy}(t) \stackrel{\text{def}}{=} \frac{1}{2}m\|\dot{\mathbf{x}}\|^2 + m\,U\bigl(\|\mathbf{x}\|\bigr) \tag{2}$$

is independent of t (where, as in part(e), $U'=u$). [Hint: show that d/dt applied to Energy(t) is zero.] [An earlier version of the homework had a $-$ instead of a $+$ in (2).]

(3) Consider the central force problem (1), where $\mathbf{x}\colon\mathbb{R}\to\mathbb{R}^2$, for fixed u,U as in Problem (2). We may write (1) as a 4-dimensional ODE by setting $\mathbf{y}=(y_1,y_2,y_3,y_4)=(\dot{x}_1,\dot{x}_2,x_1,x_2)$ and letting

$$r = \|\mathbf{x}\| = \sqrt{y_3^2+y_4^2},$$

and hence

$$\frac{d}{dt}\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} = \mathbf{f}(\mathbf{y}), \quad \text{where} \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} -u(r)y_3/(mr)\\ -u(r)y_4/(mr)\\ y_1\\ y_2\end{bmatrix}, \quad r=\sqrt{y_3^2+y_4^2}. \tag{3}$$

Notice that the energy of the system Energy(t) can therefore be written as

$$E(\mathbf{y}) = \frac{1}{2}m\|(y_1,y_2)\|^2 + m\,U\bigl(\|(y_3,y_4)\|\bigr),$$

and hence $E(\mathbf{y}(t))$ is independent of t. One way to test the accuracy of numerical (i.e., approximate) solutions to (3) is to see if the approximation to $E(\mathbf{y}(t))$ changes in time. Newton's Law of Gravitation (to predict how planets move around the sun) fixes a real constant $g>0$, and takes $U(r)=-g/r$ and so $u(r)=g/r^2$.

(a) Consider the case where $m = g = 1$, and we solve (3) subject to

(4) $$\mathbf{y}(0) = [0, 0.8, 1, 0].$$

Use MATLAB to generate points $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N$ using Euler's method

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}(\mathbf{y}_i), \quad i = 0, 1, \ldots, N-1$$

with the following values of h, N:

(i) First take $h = 0.1$ and $N = 600$. (Hence you are approximating $y(t)$ for $0 \le t \le iN = 60$.) What is $E(\mathbf{y}_0)$, and $E(\mathbf{y}_N)$? Does $E(\mathbf{y}_i)$ always increase, always decrease, or does it fluctuate in both directions? Does the approximate ellipse that $\mathbf{y}_i$ traces out (in its 3rd and 4th components, which approximates $\mathbf{x}(t)$) seem to get larger in time or get smaller (or is it roughly the same)?

(ii) Next take $h = 0.01$ and $N = 6000$. (Hence you are still approximating $y(t)$ for $0 \le t \le 60$, but presumably the approximation is getting better.) Same questions.

(iii) Same questions with $h = 0.001$ and $N = 60000$.

(b) Same questions with $h = 0.1$ and $N = 600$, but this time use the (explicit) trapezoidal method.

[Hint: you are welcome to use the program `Euler_Central_Force.m` that I'm supplying; and can observe Euler's method in the above three cases by typing each of the three lines:

```
Euler_Central_Force(1,1,[0,.8,1,0],0.1,600,.05,1)
Euler_Central_Force(1,1,[0,.8,1,0],0.01,6000,.05,10)
Euler_Central_Force(1,1,[0,.8,1,0],0.0001,600000,.05,1000)
```

For the trapezoid rule, you'll probably want to modify `Euler_Central_Force.m` by replacing the line:

```
for i=1:N
  yvals(i+1,:) = yvals(i,:) + h * f( yvals(i,:) );
end
```

with something like:

```
for i=1:N
  Y = yvals(i,:) + h * f( yvals(i,:) );
  yvals(i+1,:) = yvals(i,:) + (h/2) * ( f( yvals(i,:) ) + f( Y ));
end
```

However, you may likely be able to do a better job by writing your own code. Also, you can likely answer these questions without plotting anything; plotting just makes the answers easier to see.]

(4) Consider the recurrence $x_{n+2} - x_n = 0$.

(a) Write the general solution as $x_n = c_1 r_1^n + c_2 r_2^n$ for some values of r_1, r_2.

(b) Given the initial conditions $x_0 = 5$ and $x_1 = 7$, solve for c_1, c_2, and use these values to get a formula for x_n for any n.

(c) Given the initial conditions $x_0 = 5$ and $x_1 = 7$, what are the values of $x_9, x_{10}, x_{11}, x_{12}$? Was the above method of solving for c_1, c_2 the quickest way to determine these values?

(d) Write the recurrence as $\mathbf{y}_{n+1} = A\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix};$$

what is A? What are the values of $A^9, A^{10}, A^{11}, A^{12}$?

(5) Consider the recurrence $x_{n+2} - 4x_{n+1} + 4x_n = 0$ for all $n \in \mathbb{Z}$, with x_0, x_1 given (but arbitrary).

(a) Since the general solution of this recurrence (seen in class) is $x_n = c_1 2^n + c_2 n 2^n$, solve for c_0, c_1 in terms of x_0, x_1. Use this to get a formula for x_n for given x_0, x_1.

(b) Write the recurrence as $\mathbf{y}_{n+1} = A\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix};$$

what is A? Using the fact that $2I - A = N$ where $N^2 = 0$ (the zero matrix), derive a formula for A^n for any $n = 0, 1, 2, \ldots$. Use this to derive a formula for x_n in terms of x_0, x_1.

(c) For a small but nonzero ϵ, consider the recurrence

$$(\sigma - 2)(\sigma - 2 - \epsilon)(x_n) = 0,$$

i.e.,

$$x_{n+2} - (4+\epsilon)x_{n+1} + (4 + 2\epsilon)x_n = 0.$$

We can write this as a recurrence $\mathbf{y}_{n+1} = A(\epsilon)\mathbf{y}_n$, where

$$\mathbf{y}_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix},$$

and

$$A = \begin{bmatrix} 4+\epsilon & -4-2\epsilon \\ 1 & 0 \end{bmatrix}$$

From the general theory of recurrences, we know that A has an eigenvalue 2 with eigenvector $[2; 1]$, and an eigenvalue $2+\epsilon$ with eigenvector $[2+\epsilon; 1]$, and hence

(5) $$A(\epsilon) = S(\epsilon) \begin{bmatrix} 2 & 0 \\ 0 & 2+\epsilon \end{bmatrix} \bigl(S(\epsilon)\bigr)^{-1}, \quad \text{where} \quad S = \begin{bmatrix} 2 & 2+\epsilon \\ 1 & 1 \end{bmatrix}$$

where

$$S = \begin{bmatrix} 2 & 2+\epsilon \\ 1 & 1 \end{bmatrix}$$

For any n, find

$$\lim_{\epsilon \to 0} \bigl(A(\epsilon)\bigr)^n$$

using (5). Show that it agrees with the matrix in part (b). [Hint: the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

shows that

$$\bigl(S(\epsilon)\bigr)^{-1} = \frac{1}{-\epsilon} \begin{bmatrix} 1 & -2-\epsilon \\ -1 & 2 \end{bmatrix},$$

it suffices to write

$$S(e) = M_1 + \epsilon M_2, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2+\epsilon \end{bmatrix}^n = M_3 + \epsilon M_4 + O(\epsilon^2), \quad \begin{bmatrix} 1 & -2-\epsilon \\ -1 & 2 \end{bmatrix} = M_5 + \epsilon M_6,$$

and to consider the constant and order ϵ terms of

$$(M_1 + \epsilon M_2)(M_3 + \epsilon M_4)(M_5 + \epsilon M_6)$$

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