GROUP HOMEWORK 4, CPSC 303, SPRING 2024

JOEL FRIEDMAN

Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Recall the (forward) Euler's method (see class notes or [A&G], page 485, and the (explicit) trapazoidal method (class notes or [A&G], page 494), which solves $y' = f(t, y)$ subject to $y(t_0) = y_0$ (1) choosing a small $h > 0$ and setting $t_i = t_0 + ih$ (as usual), and (2) takes y_i (an approximation of $y(t_i)$) to produce y_{i+1}, for $i = 0, 1, 2, \ldots$ via the formula the two formulas:

$$Y_i = y_i + hf(t_i, y_i), \quad y_{i+1} = y_i + h\frac{f(t_i, y_i) + f(t_{i+1}, Y_i)}{2}$$

(hence Y_i is what Euler's method gives for y_{i+1}).

(a) Consider the trapezoidal method for $y' = y$ with $y(t_0) = y_0$. Write down a formula for y_{i+1} of the trapezoidal method in terms of y_i.
(b) Say that we want to approximate $y(1)$ given $y(0) = 1$ (i.e., $t_0 = 0$, $y_0 = 1$). Choose a natural N (which we think of as large), set $h = 1/N$; show that y_N in the trapezoidal method approximatino of $y(1)$ equals

$$y_{N,\text{trap}} = \left(1 + \frac{1}{N} + \frac{1}{2N^2}\right)^N.$$

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(c) Show that Euler's method for the same ODE and initial condition gives $y_{N,\mathrm{Eul}} = (1 + 1/N)^N$.
(d) Recall that actual solution to $y' = y$ with $y(0) = 1$ has $y(1) = e$. Show that $\ln(y_{N,\mathrm{trap}})$ is closer to the actual value of $\ln(y(1))$ (namely $\ln(e) = 1$) than $\ln(y_{N,\mathrm{Eul}})$, as $N \to \infty$ (i.e., for all N sufficiently large). [It may be helpful to recall that $\ln(1 + \epsilon) = \epsilon - \epsilon^2/2 + O(\epsilon^3)$ for $|\epsilon|$ small.]

(3) In this exercise we will solve the ODE $y' - 2y = t^2$.
(a) Show that there are constants a, b, c such that $y(t) = at^2 + bt + c$ is a solution to $y' - 2y = t^2$.
(b) Show that if $y' - 2y = t^2$ and $z' - 2z = 0$, then $(y+z)' - 2(y+z) = t^2$.
(c) Show that if $y' - 2y = t^2$ and $(y+z)' - 2(y+z) = t^2$, then $z' - 2z = 0$.
(d) Solve for all $z' - 2z = 0$ (which is called the "homogeneous form of the equation $y' - 2y = t^2$"), and use this to write down the general solution to $y' - 2y = t^2$.
(e) Show that for any t_0, y_0, your general solution in part (d) has a unique y such that $y(t_0) = y_0$.

(4) In this exercise we will solve the recurrence relation $x_{n+1} - 2x_n = n^2$ for all $n \in \mathbb{Z}$.
(a) Show that there are constants a, b, c such that $x_n = an^2 + bn + c$ is a solution to $x_{n+1} - 2x_n = n^2$.
(b) What is the general solution to $x_{n+1} - 2x_n = 0$?
(c) Use the above two parts to write a general solution to the recurrence equation $x_{n+1} - 2x_n = n^2$, and explain your reasoning in term of Problem (2) above and the appropriate notation of the "homogeneous form of the recurrence $x_{n+1} - 2x_n = n^2$.

(5) (a) At what value of n does MATLAB declare $(1/2)^n$ to be 0? There are a number of ways of doing this, but one way is to examine the values of x generated by:

```
clear
for n=1:1100, x{n}=(1/2)^n; end

x
```

(you may need the extra blank line above if you copy and paste).
(b) Find the general solution to the recurrence $x_{n+2} = (3/2)x_{n+1} - (1/2)x_n$
(c) Find the special case of this recurrence subject to the initial conditions $x_1 = 1$ and $x_2 = 1/2$.
(d) Use MATLAB to compute the solution of this recurrence subject to $x_1 = 1$ and $x_2 = 1/2$ for $x_1, x_2, \ldots, x_{100}$ via:

```
clear
x{1} = 1;
x{2} = 1/2;
for n=1:98, x{n+2} = (3/2)*x{n+1}-(1/2)*x{n}; end

x
```

What does MATLAB report for the values of

```
x{100} * 2^99
x{100} * 2^99 - 1
```

(e) Now run the same code to generate $x_1, x_2, \ldots, x_{1200}$ using

```
clear
x{1} = 1;
x{2} = 1/2;
for n=1:1198, x{n+2} = (3/2)*x{n+1}-(1/2)*x{n}; end

x
```

Answer the following:

(i) Does MATLAB report $x_n = 0$ for some value of $n \leq 1200$?

(ii) What is the smallest value of n_0 such that the values that MATLAB reports for $(1/2)^{n_0-1}$ and $x\{n_0\}$ are not equal?

(iii) For the value of n_0, by examining the values of $(3/2) * x\{n_0 - 1\}$ and $(1/2) * x\{n_0 - 2\}$, can you see where there an error in precision occurs?

(iv) What repeating pattern do you see in the values of $x\{n\}$ for $n \geq n_0$? [Hint: It may be simpler to report this pattern in multiples of $(1/2)^{1074}$.] [Warning: if you type something like

```
for n=1070:1100, {n,x{n},x{n}*2^1074}, end
```

then the last cell value will be `Inf`, since 2^{1074} will yield `Inf`. However, if you type:

```
for n=1070:1100, {n,x{n},(x{n}*2^900)*2^174}, end
```

then you'll get the answers you want.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA.

E-mail address: jf@cs.ubc.ca

URL: http://www.cs.ubc.ca/~jf