GROUP HOMEWORK 3, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Using the expansion $e^{Ah} = I + hA + (1/2)(hA)^2 + O(h^3)$ for $|h|$ small (i.e., near 0) (where the constant in $O(h^3)$ depends on A), show that for any fixed square matrices A, B of the same dimensions,

$$e^{(A+B)h} - e^{Ah}e^{Bh} = (1/2)(BA - AB)h^2 + O(h^3)$$

for $|h|$ small.

(3) Recall that Euler's method for solving $y' = f(y)$ subject to $y(t_0) = y_0$ involves (page 485 of [A&G], where a, c there are t_0, y_0 here) (1) choosing an $h > 0$, (2) setting $t_i = t_0 + ih$, and (3) approximating $y(t_i)$ as y_i using the formula

$$y_{i+1} = y_i + hf(y_i).$$

Now consider the ODE where $f(y) = |y|^{1/2}$.

(a) Say that for some $i \geq 1$, $y_{i+1} = 0$. Show that y_i is either 0 or $-h^2$.
(b) Say that for some $i \geq 1$, $y_{i+1} < 0$. Show that:
  (i) $y_i < 0$; and

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(ii) using part (i), setting $x = \sqrt{-y_i}$, i.e., x is the positive square root of $-y_i$, show that

$$x = \frac{h + \sqrt{h^2 - 4y_{i+1}}}{2},$$

(where $\sqrt{h^2 - 4y_{i+1}}$ refers to the positive square root) and hence

$$y_i = -\left(\frac{h + \sqrt{h^2 - 4y_{i+1}}}{2}\right)^2.$$

(c) Show that if for some $i \geq 1$, $y_{i+1} < 0$ and $y_{i+1} = -uh^2$ for a real $u > 0$, then

$$y_i = -h^2 g(u) \quad \text{where} \quad g(u) = \frac{1 + 2u + \sqrt{1 + 4u}}{2}.$$

(4) Consider the ODE $y' = |y|^{1/2}$, with subject to $y(0) = y_0$ and arbitrary $h > 0$.

(a) Using the results of the previous exercise, find a value of $y_0 < 0$ (as a function of h) such that (in exact arithmetic) $y_1 = 0$, and therefore $y_2 = 0$, $y_3 = 0$, etc.

(b) Using the results of the previous exercise, find a value of $y_0 < 0$ (as a function of h) such that (in exact arithmetic) $y_1 < 0$ and $y_2 = y_3 = \cdots = 0$.

(c) Use the code from Homework 2 called `chaotic_sqrt.m`, or implement your own Euler's method solver for $y' = |y|^{1/2}$. What does MATLAB report for $y(2)$ on the initial condition $y(0) = y_0$ where $h = 2/N$ for the values: $N = 2, 4, 16, 64, 65, 63, 99, 100, 101$ and $y_0 = -(2/N)^2 = -h^2$? Make a table of computed values of $y(2)$. Which values of N above yields something likely due to roundoff or truncation error in finite precision? (Therefore you might want to type something like `chaotic_sqrt(0,2,63,-4/(63^2));` into MATLAB for $N = 63$.)

(d) Use the code from Homework 2 called `chaotic_sqrt.m`, or implement your own Euler's method solver for $y' = |y|^{1/2}$. What does MATLAB report for $y(2)$ on the initial condition $y(0) = y_0$ where $h = 2/N$ for the values: $N = 7, 8, 9, 10, 100, 101, 10^6, 10^6 + 1$ and for

$$y_0 = -h^2\left(\frac{3 + \sqrt{5}}{2}\right)?$$

Type this same expression into MATLAB as written above (e.g., don't write down an approximation for $\sqrt{5}$). (Therefore you might want to type something like `chaotic_sqrt(0,2,100,-(4/(100^2))*(3+sqrt(5))/2);` into MATLAB for $N = 100$.) Which values of N above yields something likely due to roundoff or truncation error in finite precision?

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