GROUP HOMEWORK 2, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) Familiarize yourselves with basic MATLAB syntax, and make sure you understand what each line in the file `start_here.txt` is doing, and what the commands in `exponential_of_a_matrix.txt` are doing. Answer the following questions with MATLAB (but just write down the answer).

(a) Let
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
What is the largest integer n such that each entry of $e^A - \sum_{i=0}^{15} A^i/i!$ is of absolute value at most 10^{-n}?

(b) Same question for
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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(3) Create the files `apple.m`, `apple_bad.m`, `apple_worse.m`, `apple_quiet.m` and see how they are implementing Euler's method to solve $y' = 2y$ subject to $y(1) = 3$, in order to find $y(2)$. (The files `apple_bad.m`, `apple_worse.m` will produce error messages; they are just there as a cautionary note.) You might also have a look at `chaotic_sqrt.m`.

(a) Use Euler's method to solve $y' = |y|^{1/2}$ subject to $y(t_0) = y_0$ to find the value of $y(t_{\mathrm{end}})$, where $t_0 = -2$, $y_0 = -1$, and $t_{\mathrm{end}} = 2$. Use step size $h = (t_{\mathrm{end}} - t_0)/N$, where $N = 1000$ and $N = 100,000$. What values do you get?

(b) Same question, but with $y_0 = 0$, $t_0 = 0$, and $t_{\mathrm{end}} = 2$. Use $N = 10000$.

(c) Same question, but with $y_0 = 10^{-20}$.

(d) Same question, but with $y_0 = 10^{-40}$.

(e) How do you explain the difference between parts (c,d) and part (b)?

(4) If $y \colon \mathbb{R} \to \mathbb{R}$ is a function and $T \in \mathbb{R}$, then the *translation of y by T*, denoted $\mathrm{Trans}_T(y)$, refers to the function z given by

$$\forall t \in \mathbb{R}, \quad z(t) = y(t - T)$$

(or, equivalently, $z(t + T) = y(t)$) (hence $z(T) = y(0)$, $z(T + 1) = y(1)$, etc.). Similarly, the *time reversal of y at time T*, denoted $\mathrm{Reverse}_T(y)$, refers to the function z given by

$$\forall t \in \mathbb{R}, \quad z(t) = y(2T - t)$$

(hence $z(T) = y(T)$, and $z(T + a) = y(T - a)$).

(a) If $T_1, T_2 \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Trans}_{T_1}\bigl(\mathrm{Trans}_{T_2} y\bigr)$$

in simpler terms?

(b) If $T \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Reverse}_T\bigl(\mathrm{Reverse}_T(y)\bigr)$$

in simpler terms?

(c) If $T_1, T_2 \in \mathbb{R}$ and y is any function, what is

$$\mathrm{Reverse}_{T_1}\bigl(\mathrm{Reverse}_{T_2}(y)\bigr)$$

in simpler terms?

(d) Show that if for some function $f \colon \mathbb{R} \to \mathbb{R}$, y satisfies the ODE $y' = f(y)$ globally (meaning $y'(t) = f(y(t))$ for all $t \in \mathbb{R}$), then $z = \mathrm{Trans}_T(y)$ satisfies the same ODE, i.e., $z' = f(z)$ (globally).

(e) Show directly that if $y(t) = e^{At}$ for some $A \in \mathbb{R}$, and if z satisfies the ODE $z' = Az$ with $z(t) > 0$ for some $t \in \mathbb{R}$, then z is a translation of y, **PROVIDED THAT** $A \neq 0$.

(f) If similarly $y' = f(y)$ globally, then $z = \mathrm{Reverse}_T(y)$ satisfies the ODE $z' = -f(z)$.

(g) If similarly $y'' = f(y)$ globally, then $z = \mathrm{Reverse}_T(y)$ satisfies the ODE $z'' = f(z)$.

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