GROUP HOMEWORK 1, CPSC 303, SPRING 2024

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**
(5) At times we may only grade part of the homework set. The number of points per problem (at times indicated) may be changed.

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(1) (0 to -6 points) Who are your group members? Please print if writing by hand. [See (4) above.]

(2) (2 points)
   (a) Show that the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

   has the unique solution

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 4/2 \\ -1/2 \end{bmatrix}$$

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(b) Say that $f\colon \mathbb{R} \to \mathbb{R}$ has $f'''(x)$ existing for all x. Say that $x_0, h \in \mathbb{R}$, and that $|f'''(\xi)| \le M_3$ for all ξ bewteen x_0 and $x_0 + 2h$. Use the fact that

$$\begin{aligned} f(x_0) &= f(x_0) \\ f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)M_3 \\ f(x_0+2h) &= f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2} f''(x_0) + O(h^3)M_3 \end{aligned}$$

to find a value of c_0, c_1, c_2 such that

$$c_0 f(x_0) + c_1 f(x_0+h) + c_2 f(x_0+2h) = hf'(x_0) + O(h^3)M_3.$$

(c) To which formula on page 411 (Section 14.1) of [A&G] are parts (a) and (b) related? Explain.

(d) What is the significance of the solution of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

to approximating $f'(x_0)$? [You don't have to solve this system, just state what you can do with the solution c_0, c_1, c_2, c_3.]

(3) (2 points) Consider an ODE $y' = f(t,y)$, where as in [A&G], $y = y(t)$, and y' refers to dy/dt. Say that $f(t,y)$ is of the special form $f(t,y) = h(t)g(y)$, where g is differentiable function and h is continuous. Then the ODE

$$y' = dy/dt = h(t)g(y)$$

is called a *separable differential equation*, and it can be solved by writing

$$\frac{dy}{g(y)} = h(t)\,dt,$$

and taking indefinite integrals of both sides. See Section 2.4 (Separable ODE's) of UBC's Calculus 2 Textbook for details, including Example 2.4.2 there, where they solve the equation $y' = y^2$ (in this textbook, y' refers to dy/dx, as is common in math books).

(a) Solve the ODE $y' = y^3$ (here $y = y(t)$ and y' refers to dy/dt) in the same manner as $y' = y^2$ is solved in in general form.
(b) Solve $y' = y^3$ for the initial condition $y(1) = 1$.
(c) Solve $y' = y^4$ for the initial condition $y(1) = 1$.
(d) Let $y(t)$ be as in part (b); for $t \ge 1$, when does $y(t)$ fail to exist, i.e., for which $T > 1$ does $y(t) \to \infty$ as $t \to T$?
(e) Same question for part (c).

(4) (2 points) Let $y(t) = (3-2t)^{-1/2}$, $z(t) = (4-3t)^{-1/3}$.

(a) Examine a plot of $y(t)$ and $z(t)$ for $1 \le t < 4/3$. Is one of these functions larger than the other in the entire interval (as far as the plot shows)? (Here a simple answer will do. You might type
`plot (3-2t)^(-1/2) and (4-3t)^(-1/3)`

into Google, or someting like that.)

(b) Show that $y' = y^3$ and $z' = z^4$ and that $y(1) = z(1) = 1$.

(c) Show that $y'(1) = z'(1) = 1$.

(d) Show that $y''(1) = 3$ and $z''(1) = 4$. [Hint: differentiate both sides of $y' = y^3$; similarly for z.]

(e) Show that for $h > 0$ and h sufficiently small we have $y(1+h) < z(1+h)$. [Hint: let $u(t) = z(t) - y(t)$; what are the values of $u(1), u'(1), u''(1)$?]

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