Midterm Prep Solutions

(1) (a) false : the solution is $y(t) = e^{Ay}\vec{v}$, any $\vec{v} \in \mathbb{R}^m$

(b) True: given $y(t_0) = \vec{y}_0$, the unique solution is

$y(t) = e^{A(t-t_0)}\vec{y}_0$; here $t_0 = 5$.

(c) False. See, for example, HW7, Problem 6:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \vec{c} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} y_0 \\ (y_1 - y_0)/\varepsilon \end{bmatrix}$$

but $K_\infty \begin{bmatrix} 1 & 2 \\ 1 & 2+\varepsilon \end{bmatrix} \neq K_\infty \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(d) False (e) True : (see HW7, Prob 4, and class notes)

If $A\vec{x} = \vec{b}$, $A\hat{x} = \hat{b}$, then

$\mathrm{RelErr}_\infty(\hat{x}, \vec{x}) \le K_\infty(A)\, \mathrm{RelErr}_\infty(\hat{b}, \vec{b})$, with

equality iff $\vec{b}_{error} = \hat{b} - \vec{b}$ satisfies

$$\frac{\|A^{-1}\vec{b}_{error}\|_\infty}{\|\vec{b}_{error}\|_\infty} = \|A^{-1}\|_\infty \quad \text{and} \quad \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} = \|A\|.$$

(f) true (since $f(y) = y^2$ is differentiable at y_0)

(g) false (see HW1, Prob 3 for (g) and (f))

(h) false (i) true

The general solution to $y' = |y|^{1/2}$ is for $a \le b$

$$y(t) = \begin{cases} -\frac{1}{4}(t-a)^2 & t \le a \\ 0 & a \le t \le b \\ -\frac{1}{4}(t-b)^2 & t \ge b \end{cases}$$

(j) false (k) true ; $z'(t) = \frac{d}{dt}\left(y(-t)\right) = y'(-t)(-1) = y^2(-t)(-1)$

$= z^2(t)(-1)$ (see also HW 2, Prob 4(f))

(l) true: similarly (see also HW 2, Prob 4(g))

(m) true: see HW 7, Probs (1) and (2)

(2) $f(x) = f(x)$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x+3h) = f(x) + 3h f'(x) + \frac{(3h)^2}{2} f''(x) + O(h^3)$$

so $c_0 f(x) + c_1 f(x+h) + c_2 f(x+3h)$

$$= (c_0 + c_1 + c_2) f(x) + (c_1 + 3c_2) h f'(x)$$

$$+ \left(c_1/2 + 9c_2/2 \right) h^2 f''(x) + O(h^3)$$

So if

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1/2 & 9/2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

then

$$\frac{c_0 f(x) + c_1 f(x+h) + c_2 f(x+3h)}{h} = f'(x) + O(h^2)$$

(Compare HW 1, Problem 2)

(3) & (4) Compare HW 2, Prob(4):

$$\frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow \frac{-1}{y} = t + C$$

$$\Rightarrow y(t) = \frac{-1}{t+C}.$$

For (3): $\frac{-1}{y} = t + C \Rightarrow \frac{-1}{1} = 3 + C$, $C = -1-3 = -4$

So $y(t) = \frac{1}{4-t} \Rightarrow$ as $t \to 4 = T$, $y(t) \to \infty$.

For (4): $\frac{-1}{y} = t + C \Rightarrow \frac{-1}{1} = 0 + C \Rightarrow C = -1$

so $y(t) = \frac{1}{1-t} \Rightarrow$ as $t \to 1$, $y(t) \to \infty$.

(5) (a) $$\dot{\vec{y}} = \frac{d}{dt}\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ x \end{bmatrix}$$

So $\dot{\vec{y}} = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) See HW 5, Prob (2): (here $\vec{x}(t) = x(t)$ is one-dimensional)

so $$m\ddot{x} = -m\, u(\|x\|)\frac{x}{\|x\|}$$

where $u(\|x\|) = \|x\|$, so $u(r) = r$

and $U'(r) = u(r)$ where $U(r) = \frac{1}{2}r^2 + C$

So
$$\text{Energy}(t) = \frac{1}{2}m\|\dot{x}\|^2 + m\,U(\|x\|)$$
$$= \frac{1}{2}m|\dot{x}|^2 + m\left(\frac{1}{2}|x|^2 + C\right)$$

(where C is an arbitrary constant).

(b) Using a method in class, 03_08 in 2024,

$m\ddot{x} = mx$, so $m\ddot{x}\dot{x} = mx\dot{x}$, so

$\frac{1}{2}m(\dot{x})^2 = \frac{1}{2}m(x)^2 + C$ so

$$\text{Energy} = \frac{1}{2}m(\dot{x})^2 - \frac{1}{2}mx^2 + \text{arbitrary constant}$$

(c) So $x(t) = c_1e^t + c_2e^{-t}$ are both solutions,

so $1 = x(0) = c_1 + c_2$, $0 = \dot{x}(0)$ and

$\dot{x}(t) = c_1 e^t - c_2 e^{-t}$ so $0 = \dot{x}(0) = c_1 - c_2$.

So we solve

$$\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 \Rightarrow c_1 = c_2 = \tfrac{1}{2}.$$

So

$$x(t) = \frac{1}{2} e^t + \frac{1}{2} e^{-t}$$

(6) Similar to (5), $\dot{\vec{y}} = A\vec{y}$ where

$$A = \begin{bmatrix} 0 & -C \\ 1 & 0 \end{bmatrix}$$

and

$$\text{Energy}(t) = \frac{1}{2} m (\dot{x})^2 + \frac{1}{2} m C x^2$$

(7) See HW 4, Problem (5)

(a) $x_n = c_1 + c_2 \left(\frac{1}{2}\right)^n$

(b) $1 = x_0 = c_1 + c_2$ and $\frac{1}{4} = x_1 = c_1 + \frac{c_2}{2}$

so

$$\begin{array}{l} c_1 + c_2 = 1 \\ c_1 + c_2/2 = 1/4 \end{array} \Rightarrow c_2/2 = 3/4 \Rightarrow c_2 = 3/2$$

$$c_1 = -1/2$$

so $x_n = \frac{-1}{2} + 3/2 \left(1/2\right)^n$

So $x_n \to -1/2$ in exact arithmetic,

and in MATLAB $x_n \longrightarrow -1/2$ to

within a factor of (1 ± 2^{-52}) or so,

since $-1/2$ is a normal number. So

MATLAB will report $x_n = 1/2$ for large

n, to within the limits of double precision.

[However, it is not clear that $x_n - 1/2$

will report 0 or $(-1/2) \cdot (\pm 2^{-52}) \ldots$]

(c) HW 4, Problem(5): $x_n = \left(\frac{1}{2}\right)^n$, but

we know this will cycle (with period 8)

around the values $2^{-1074} \cdot n$ with

$n = 1, 2, 3$.

(8) Compare HW 4, Prob(4)

Homogeneous recurrence is

$x_{n+1} - 2x_n = 0$, solution $x_n = 2^n c$.

Try $x_n = an + b$ for particular solution

to $\underbrace{x_{n+1} - 2x_n}_{\underbrace{a(n+1) + b - 2(an+b)}_{n\cdot(-a) + (a-b)}} = n + 3$ $\Rightarrow$ $-a = 1$, $a - b = 3$ $\Downarrow$ $a = -1$, $b = a - 3 = -4$

So $x_n = -n - 4$ is particular solution.

So general solution is $x_n = c2^n - n - 4$.

(9) Compare HW 4, Problem (3).

Homogeneous ODE has solution $y(t) = e^{2t}c$,

Particular solution to $y(t) = at + b$

gives $\underbrace{y' - 2y}_{a - 2(at+b)} = t + 3$ $\Bigg\}$ $a = \frac{-1}{2}$, $b = \frac{a-3}{2} = -7/4$

So $\quad y(t) = e^{2t} c + \left(\frac{-1}{2} t - \frac{7}{4} \right)$

(10) $\quad \|A\|_\infty = \max(1+4, 1+4+\varepsilon), \; (\varepsilon = 10^{-4})$

$$= 5 + \varepsilon = 5 + 10^{-4}$$

$$A^{-1} = \frac{1}{\varepsilon} \begin{bmatrix} 4+\varepsilon & -4 \\ -1 & 1 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \frac{1}{\varepsilon}(8+\varepsilon)$$

So $\quad K(A) = \dfrac{(5+\varepsilon)(8+\varepsilon)}{\varepsilon} \approx 10^4(40 + 13\varepsilon + \varepsilon^2)$

$$\approx 4 \cdot 10^5$$

So $\quad \mathrm{RelErr}_\infty(\hat{x}, \vec{x}) \leq 4 \cdot 10^5 \; \mathrm{RelErr}_\infty(\hat{b}, \vec{b})$

(roughly)

$$\approx 4 \cdot 10^5 \cdot 10^{-8} = 4 \cdot 10^{-3}$$

So $\vec{x}$ is known to relative error

at most $\quad 4 \cdot 10^{-3}$.

(11) Compare HW 4, Prob(2)

(a) $y(2) = e^{5(t-t_0)} y_0 = e^{5t} \cdot 4$

(b) $y_{i+1} = y_i + h\,5y_i = (1+h5)\,y_i$

so, by induction

$$y_i = (1+5h)^i\, y_0$$

and y_i is an approximation to $y(t_0+ih)$.

With $h = 1/n$ and $t_0 = 0$, y_i approximates

$y(2)$ for $t_0 + ih = 2$, i.e, $i \cdot \frac{1}{n} = 2$,

i.e. $i = 2n$. So

$$y_{2n} = (1+5h)^{2n}\, y_0$$

$$= (1 + 5/n)^{2n}\, 4$$

approximates $y(2)$ with Euler's method, $h = \frac{1}{n}$.

(c) $\log\left((1+5/n)^{2n} \cdot 4 \right)$

$= 2n \log(1+5/n) + \log 4$

$= 2n \left(5/n + O(1/n^2) \right) + \log 4$

$= 10 + O(1/n) + \log 4$

So $\log\left((1+5/n)^{2n} 4\right) \xrightarrow{n\to\infty} 10 + \log 4.$

and $\log(y(2)) = \log(e^{10} \cdot 4)$

$= \log(e^{10}) + \log(4) = 10 + \log 4$

Hence $\lim_{n\to\infty} \left((1+5/n)^{2n} 4\right) = e^{10+\log 4} = e^{10} 4$

$= y(2)$

(12) Compare HW 4, Prob(2)

(a) $y(t) = e^{5(t-t_0)} y_0 = e^{5t} \cdot 4$

So $y(1/n) = e^{5/n} \cdot 4$

(b) For $h = 1/n$, Y_1 approximates $y(t_0+h) = y(1/n)$

and $Y_1 = Y_0 + h(5Y_0) = (1 + 5/n) Y_0 = (1 + 5/n) 4$

(c) Similarly (see HW 4, Prob (2a))

$$\tilde{Y}_1 = (1 + 5/n) 4$$

$$Y_1 = Y_0 + h \frac{(5Y_0 + 5\tilde{Y}_1)}{2}$$

$$= 4 + \frac{1}{n} \frac{5 \cdot 4 + 5 \cdot ((1 + 5/n) 4)}{2}$$

$$= 4 \left(1 + \frac{1}{n} \frac{5 + 5 + 5^2/n}{2} \right)$$

$$= 4 \left(1 + \frac{5}{n} + \frac{5^2}{2} \cdot \frac{1}{n^2} \right)$$

(d) Euler: $4(1+5/n)$

Explicit Trap: $4\left(1+5/n+\frac{5^2}{2}\frac{1}{n^2}\right)$

Exact: $y(\cdot 1/n) = e^{5/n}\cdot 4$, so

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1 Exact $= \left(1+5/n + \frac{1}{2}\, 5^2/n^2 + O(1/n^3)\right) 4$

for n large.

Hence Explicit Trap and Exact agree to $O(1/n^3)$, which is closer than Euler is to Exact (which is $O(1/n^2)$).

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2 Alternatively:

Taking logs:

$\log(\text{Euler}) = \log 4 + \log(1+5/n)$

$$= \log 4 + 5/n - \tfrac{1}{2}(5/n)^2 + O(1/n^3)$$

$\log(\text{Exp. Trap}) =$

$$\log 4 + \left(5/n + \tfrac{1}{2}\, 5^2/n^2\right) - \tfrac{1}{2}\left(5/n + \tfrac{1}{2}\, 5^2/n^2\right)^2$$

$$+ O(1/n^3)$$

$$= \log 4 + 5/n + O(1/n^3)$$

$$\log(\text{Exact}) = \log(e^{5/n} \cdot 4)$$

$$= \log 4 + 5/n$$

Hence $\log(\text{Exp. Trap})$ is closer than

$\log(\text{Euler})$ to $\log(\text{Exact})$

Note: Using $e^x = 1 + x + x^2/2 + O(x^3)$

for $|x|$ small is simpler than using

$\log(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + O(\varepsilon^3)$ for $|\varepsilon|$ small