MIDTERM PRACTICE, CPSC 303, SPRING 2024

JOEL FRIEDMAN

The following problems illustrate how some homework problems will be translated into problems on the midterm. **These problems do not cover all the homework**

(1) Answer true or false:
   (a) If A is an $m \times m$ matrix, the equation $\mathbf{y}' = A\mathbf{y}$ has a unique solution.
   (b) If A is an $m \times m$ matrix, and $\mathbf{y}_0 \in \mathbb{R}^m$, the equation $\mathbf{y}' = A\mathbf{y}$ and $\mathbf{y}(5) = \mathbf{y}_0$ has a unique solution.
   (c) If the system $A\mathbf{x} = \mathbf{b}$ is equivalent to the system $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$, then the $\mathrm{cond}_\infty(A) = \mathrm{cond}_\infty(\tilde{A})$.
   (d) If we solve $A\mathbf{x} = \mathbf{b}$ but only know $\mathbf{x}, \mathbf{b}$ approximately, then the relative error in $\mathbf{x}$, measured in the ∞-norm, equals that in $\mathbf{b}$ times $\mathrm{cond}_\infty(A)$.
   (e) If we solve $A\mathbf{x} = \mathbf{b}$ but only know $\mathbf{x}, \mathbf{b}$ approximately, then the relative error in $\mathbf{x}$, measured in the ∞-norm, is at most that in $\mathbf{b}$ times $\mathrm{cond}_\infty(A)$.
   (f) For any $t_0, y_0 \in \mathbb{R}$ with $y_0 > 0$, equation $y' = y^2$ and $y(t_0) = y_0$ has a unique solution $y(t)$ for t sufficiently close to t_0.
   (g) For any $t_0, y_0 \in \mathbb{R}$ with $y_0 > 0$, equation $y' = y^2$ and $y(t_0) = y_0$ has a unique solution $y(t)$ defined for all $t \in \mathbb{R}$.
   (h) For any $t_0, y_0 \in \mathbb{R}$ with $y_0 > 0$, equation $y' = |y|^{1/2}$ and $y(t_0) = y_0$ has a unique solution $y(t)$ defined for all $t \in \mathbb{R}$.
   (i) For any $t_0, y_0 \in \mathbb{R}$ with $y_0 > 0$, equation $y' = |y|^{1/2}$ and $y(t_0) = y_0$ has a unique solution $y(t)$ defined for all $t \geq t_0$.
   (j) If $y(t)$ is defined for t near 0 and satisfies $y' = y^2$, then $z(t) = y(-t)$ necessarily satisfies $z' = z^2$.
   (k) If $y(t)$ is defined for t near 0 and satisfies $y' = y^2$, then $z(t) = y(-t)$ necessarily satisfies $-z' = z^2$.
   (l) If $y(t)$ is defined for t near 0 and satisfies $y'' = y^2$, then $z(t) = y(-t)$ necessarily satisfies $z'' = z^2$.
   (m) Lagrange interpolation is numerically (i.e., in MATLAB) more precise than monomial interpolation at points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ when x_0, x_1, x_2 are relatively near each other.
   (n) MORE EXERCISES MAY APPEAR

(2) Divise an approximation scheme for $f'(x)$ in terms of the values $f(x), f(x+h), f(x+3h)$ that is valid to order h^2. You may use the fact that the solution to
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1/2 & 9/2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
is $c_2 = -1/6$, $c_1 = 3/2$, etc.

(3) Solve the equation $y' = y^2$ subject to the condition $y(3) = 1$. For which $T > 3$ do we have $u(t) \to \infty$ as $t \to T$?

(4) Solve the equation $y' = y^2$ subject to the condition $y(0) = 1$. For which $T > 0$ do we have $u(t) \to \infty$ as $t \to T$?

(5) Consider the equation $m\ddot{x} = mx$ for a function $x \colon \mathbb{R} \to \mathbb{R}$ and a constants $m > 0$.
  (a) Write this as a equation $\dot{\mathbf{y}} = A\mathbf{y}$ where $\mathbf{y}(t) = (\dot{x}, x)$.
  (b) Viewing $m\ddot{x} = mx$ as a central force problem, write down an expression for Energy(t) that is independent of time.
  (c) Notice that both $x(t) = e^t$ and $x(t) = e^{-t}$ are solutions to $m\ddot{x} = mx$. Write a formula for the solution to $m\ddot{x} = mx$ subject to $x(0) = 1$, $\dot{x}(0) = 0$.

(6) Consider the equation $m\ddot{x} = -Cmx$ for a function $x \colon \mathbb{R} \to \mathbb{R}$ and constants $C, m > 0$.
  (a) Write this as a equation $\dot{\mathbf{y}} = A\mathbf{y}$ where $\mathbf{y}(t) = (\dot{x}, x)$.
  (b) Viewing $m\ddot{x} = -Cmx$ as a central force problem, write down an expression for Energy(t) that is independent of time.

(7) Find the general solution to the recurrence Consider the recurrence $x_{n+2} = (3/2)x_{n+1} - (1/2)x_n$.
  (a) Find the general solution to this recurrence.
  (b) Solve this recurrence subject to $x_0 = 1$ and $x_1 = 1/4$. What do you think will happen if you use MATLAB to compute $x_2, x_3, \ldots$ for these initial conditions? Will x_n appear to have a limit as $n \to \infty$?
  (c) Solve this recurrence subject to $x_0 = 1$ and $x_1 = 1/2$. What do you think will happen if you use MATLAB to compute $x_2, x_3, \ldots$ for these initial conditions? Will x_n appear to have a limit as $n \to \infty$?

(8) Find the general solution to the recurrence $x_{n+1} = 2x_n + n + 3$.

(9) Find the general solution to the ODE $y'(t) = 2y(t) + t + 3$.

(10) Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 4+\epsilon \end{bmatrix}, \quad \text{where} \quad \epsilon = 10^{-4}.$$

Compute ∞-condition number of A. If $\mathbf{b}$ is known within a relative error of 10^{-8}, what can be said about the relative error of $\mathbf{x} = A^{-1}\mathbf{x}$?

(11) Consider the ODE $y' = 5y$ with initial value $y(0) = 4$. say that you use step size $h = 1/n$ to approximate $y(2)$.
 (a) What is the exact value of $y(2)$?
 (b) Consider Euler's method applied to this ODE with step size $h = 1/n$ for some integer n to approximate $y(2)$. Show that an exact formula for this Euler's method approximation to $y(2)$ given by

$$y(2) \approx y_{2n} = (1 + 5/n)^{2n} 4.$$

 (c) Show that as $n \to \infty$, the above approximation tends to the true solution; do this by taking logarithms, and use the formula $\log(1+\epsilon) = \epsilon + O(\epsilon^2)$ for $|\epsilon|$ small. [Since $\log(1+\epsilon) = \epsilon + O(\epsilon^2)$, all logarithms here are base e.]

(12) Consider the ODE $y' = 5y$ with initial value $y(0) = 4$. Let n be an integer.
 (a) What is the exact value of the solution of this ODE at $y(1/n)$?
 (b) What is the approximate value of this ODE given by a single step of Euler's method, i.e., with $h = 1/n$?
 (c) What is the approximate value of this ODE given by a single step of the (explicit) trapezoidal method, i.e., with $h = 1/n$?
 (d) As $n \to \infty$, which is more precise: Euler's method or the trapezoidal method? Answer this by either (1) using $e^x = 1 + x + x^2/2 + O(x^3)$ for $|x|$ small, or, (2) by taking logarithms, and use the formula $\log(1+\epsilon) = \epsilon - \epsilon^2/2 + O(\epsilon^3)$ for $|\epsilon|$ small.

(13) MORE EXERCISES MAY APPEAR

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA.
E-mail address: jf@cs.ubc.ca
URL: http://www.cs.ubc.ca/~jf