Solutions to Final Practice 2, CPSC 303, 2024

(1) If $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, we have

$$p(1) = c_0 + c_1 + c_2 + c_3$$

$$p'(1) = c_1 + 2c_2 + 3c_3$$

$$p(2) = c_0 + 2c_1 + 4c_2 + 8c_3$$

$$p'(2) = c_1 + 4c_2 + 12c_3$$

Hence given $p(1), p'(1), p(2), p'(2)$, we get a system of 4 equations with 4 unknowns. This system has a unique solution iff the homogeneous system does. So it suffices to show that there is a unique polynomial $p(x)$ as above with

$$p(1) = p'(1) = p(2) = p'(2) = 0,$$

namely $p(x) = 0$: Since $p(1) = p(2) = 0$, we have $p'(\xi)=0$ for some $1<\xi<2$.

Hence p' has ≥ 3 roots $(1,\xi,2)$; by applying Rolle's theorem to p', we have that p'' has at least 2 roots (one between 1 and ξ, another between ξ and 2); similarly p''' has at least one root. Since

$$p'''(x) = 6c_3 \quad \text{has} \geq 1 \text{ root},$$

we have $c_3=0$;

then $p(x) = c_0 + xc_1 + x^2 c_2$; since

$$p''(x) = 2c_2 \quad \text{has} \geq 2 \text{ roots},$$

$c_2=0$. Then $p(x) = c_0 + xc_1$. Since $p'(x) = c_1$ and has ≥ 3 roots, $c_1=0$.

Then $p(x)=c_0$; since $p(1)=p(2)=0$,

$c_0=0$.

Hence the only polynomial p as above is $p(x)=0$.

(2) We have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2} f''(x) + O(h^3)$$

$$f(x+3h) = f(x) + 3h f'(x) + \frac{(3h)^2}{2} f''(x) + O(h^3)$$

Hence

$$= (c_0+c_1+c_2) f(x) + (c_0+2c_1+3c_2) h f'(x)$$

$$+ (c_0+4c_1+9c_2) \frac{h^2}{2} f''(x) + O(h^3)$$

So

$$\frac{c_0 f(x+h) + c_1 f(x+2h) + c_2 f(x+3h)}{h^2} = f''(x) + O(h)$$

iff

$$c_0 + c_1 + c_2 = 0$$
$$c_0 + 2c_1 + 3c_2 = 0$$
$$c_0 + 4c_1 + 9c_2 = 2.$$

Since

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

is the transpose of a Vandermonde matrix, the above 3×3 system has a unique solution.

(3) We have

$$N_{rod,n-1} U = N_{rod,n-1} \begin{bmatrix} \sin(\pi/n) \\ \vdots \\ \sin(\pi(n-1)/n) \end{bmatrix}$$

$$= \begin{bmatrix} \sin(2\pi/n) \\ \sin(\pi/n) + \sin(3\pi/n) \\ \sin(2\pi/n) + \sin(4\pi/n) \\ \vdots \\ \sin(2\pi(n-3)/n) + \sin(2\pi(n-1)/n) \\ \sin(2\pi(n-2)/n) \end{bmatrix}$$

Since

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

(see class on 04_08)

and $\sin(0) = \sin(\pi) = 0$, we have

$$N_{rod,n-1}\, U$$

$$= \begin{bmatrix} \sin(0) \quad \sin(2\pi/n) \\ \sin(\pi/n) + \sin(3\pi/n) \\ \sin(2\pi/n) + \sin(4\pi/n) \\ \vdots \\ \sin(\pi (n-3)/n + \sin(\pi(n-1)/n) \\ \sin(\pi(n-2)/n) + \sin(\pi) \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\pi/n) \\ \sin(2\pi/n) \\ \vdots \\ \\ \sin((n-1)\pi/n) \end{bmatrix} 2\cos(\pi/n)$$

$$= U \;\; 2\cos(\pi/n)$$

Hence $2\cos(\pi/n)$ is the corresponding eigenvalue.

Note: This is a good way to conceptualize why in class on 04_08 we have that the heat equation approximation is

$$u(x, sH) \approx \left(1-2\rho+2\rho\cos(\pi h)\right)^2 u(x,0)$$

when

$$u(x,0) = \sin(\pi x).$$

(4)$^{(a)}$ We have $m\ddot{x}(t) = -u(\|x\|)\, x/\|x\|$

where $-u(\|x\|)x/\|x\| = -4x$,

So $u(\|x\|) = 4\|x\|$ so $u(r) = 4r$

So $U' = u$ for $U = 2r^2$. Hence the energy is

$$\frac{1}{2} m (\dot{x})^2 + m(2x^2).$$

(e.g. Homework 5, Problem 2).

[An alternate approach is to multiply

$$m\ddot{x} + 4mx = 0$$

by $\dot{x}$ and integrate:

$$m\ddot{x}\dot{x} + 4mx\dot{x} = 0$$

So

$$\frac{1}{2} m(\dot{x})^2 + m(2x^2) = \text{constant},$$

and the left-hand-side is the energy.]

Note: if $U(r) = 2r^2 + C$ where

C is any constant, then still $U'(r) = u(r)$.

Hence one can equally well say that the energy is

$$\frac{1}{2} m (\dot{x})^2 + m(2x^2) + C$$

for any constant, C.

(It is simplest to take $C = 0$, which we do above, and which we do for part (b))

(4 b) With

$$\text{Energy} = \frac{1}{2} m(\dot{x})^2 + m(2x^2),$$

since $x(1) = 1$ and $\dot{x}(1) = 4$,

we have

$$\text{Energy} = m\left(\left(\tfrac{1}{2}\right)16 + 2(1^2)\right) = 10m.$$

(4c) Hence for all t we have

$$10m = \text{Energy} \geq m(2x^2)$$

so $2(x(t))^2 \leq 10$ so $x(t) \leq \sqrt{5}$

5 (a) Similar to solution of Homework 4, Problem (2a).

(b) Since

$$y(h) = e^{ah} y_0 = \left(1 + ah + \frac{(ah)^2}{2} + O(h^3)\right) y_0$$

$$= \left(1 + ah + \frac{(ah)^2}{2}\right) y_0 + O(h^3),$$

any second order method must

have

y_1 = an approximation to $y(h)$

valid up to $O(h^3)$

($O(h^3)$ for a second order method),

so

y_1 must equal $y(h) + O(h^3)$

$$= \left(\left(1 + ah + \frac{(ah)^2}{2} \right) y_0 + O(h^3) \right) + O(h^3)$$

$$= \left(1 + ah + \frac{(ah)^2}{2} \right) y_0 + O(h^3)$$

(6)(a) Hence

$$y_1' = -300 y_1 , \quad y_1(0) = 2$$

$$y_2' = y_2 \quad , \quad y_2(0) = 3$$

So $$y_1(t) = e^{-300t}\, 2$$

$$y_2(t) = e^{t}\, 3$$

(or $\vec{y}(t) = (e^{-300t} 2, e^{t} 3)$).

(6b)

$$\vec{y}_{i+1} = (I + hA)\, \vec{y}_i$$

$$= \left(I + \frac{1}{100}\begin{bmatrix} -300 & 0 \\ 0 & 1 \end{bmatrix}\right) \vec{y}_i$$

$$= \begin{bmatrix} 1-3 & 0 \\ 0 & 1+\frac{1}{100} \end{bmatrix} \vec{y}_i$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 1+\frac{1}{100} \end{bmatrix} \vec{y}_i$$

So

$$\vec{Y}_i = \begin{bmatrix} -2 & 0 \\ 0 & 1+\frac{1}{100} \end{bmatrix}^i \vec{Y}_0 = \begin{bmatrix} (-2)^i \, 2 \\ \left(1+\frac{1}{100}\right)^i 3 \end{bmatrix}$$

(c) The first component is $(-2)^i 2$, which does not decrease.

(d) For $\vec{Y}_{i+1} = \vec{Y}_i + h f(\vec{Y}_{i+1})$

we have

$$\vec{Y}_{i+1} = \vec{Y}_i + h \begin{bmatrix} -300 & 0 \\ 0 & 1 \end{bmatrix} \vec{Y}_{i+1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{Y}_{i+1} = \vec{Y}_i + \begin{bmatrix} -3 & 0 \\ 0 & 1/100 \end{bmatrix} \vec{Y}_{i+1}$$

So

$$\begin{bmatrix} 4 & 0 \\ 0 & 1-1/100 \end{bmatrix} \vec{Y}_{i+1} = \vec{Y}_i$$

so

$$\vec{y}_{i+1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 100/99 \end{bmatrix} \vec{y}_i$$

So

$$\vec{y}_i = \begin{bmatrix} 1/4 & 0 \\ 0 & 100/99 \end{bmatrix}^i \vec{y}_0$$

$$= \begin{bmatrix} (1/4)^i \, 2 \\ (100/99)^i \, 3 \end{bmatrix}.$$

So for Backward Euler's method, the first component is $(1/4)^i \, 2$, which does decrease in i.

[Note! The above is an example of a "stiff ODE", where backward

Euler does better than forward Euler.]