Final Practice, Solutions

(1) (a) False: Since $m\ddot{x}$ isn't generally 0, $m\dot{x}$ depends on time.

(b) True; see HW 5, Problem 2

(c) True; true of — more generally — $y'' = f(y)$ (HW 2, Problem 4 (g))

(d) True; HW 6, Problem 3(c)

(e) False — you can scale any $\vec{x}$ s.t.

$\|Ax\|_\infty = \|A\| \, \|\vec{x}\|_\infty$ and get another example

(e.g. if $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ works, then so does $\begin{bmatrix} a \\ a \end{bmatrix}$

for any $a \in \mathbb{R}$)

(f) False — generally you have to solve an $(n+1) \times (n+1)$ system, which takes roughly $O(n^3)$ flops.

[Theoretically one can do this in some $O(n^{2.36})$ flops, but not in practice...]

(g) True; each $L_j(x)$ takes $O(n)$ flops, So $\sum_{j=0}^{n} y_j L_j(x)$ takes $O(n^2)$ flops

(h) True! see [A&G], bottom page 308.

Idea: compute

(1) $f[x_i]$, $i=0,\dots,n$	(2) $f[x_{i-1},x_i]$, $i=1,\dots,n$ as $\dfrac{f[x_i]-f[x_{i-1}]}{x_i-x_{i-1}}$	(3) $f[x_{i-2},x_{i-1},x_i]$, $i=2,\dots,n$ as $\dfrac{f[x_{i-1},x_i]-f[x_{i-2},x_{i-1}]}{x_i-x_{i-2}}$

(4) etc. ..., until last stage (n+1)

We have $n+1$ stages, where j^{th} stage takes $O(n+2-j)$ flops.

(i) False: $f[x_0,\dots,x_n]$ is independent of the order.

(2) We have n polynomials of degree 3, for $4n$ parameters. Each polynomial

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i\,(x-x_i)^3$$

has 2 given values (at $x = x_i$ and $x = x_{i+1}$), for $2n$ equations. Insisting on continuous first and second derivative at $x_1, \dots, x_{n-1}$ gives $2(n-2)$ equations. Total number of equations $= 2n + 2(n-1) = 4n-2$.

3(a) Since $S_0'(x) = b_0 + c_0 2(x-x_0) + d_0 3(x-x_0)^2$, we have $S_0'(x_0) = b_0$. Hence $b_0 = S_0'(x_0)$ $= v'(x_0) = f'(A)$; hence $b_0 = f'(A)$.

(b) $b_i = f[x_i, x_{i+1}] - \frac{h_i}{3}(2c_i + c_{i+1})$

implies

$$f'(A) = b_0 = f[x_0, x_1] - \frac{h}{3}(2c_0 + c_1)$$

Hence

$$2c_0 + c_1 = \frac{f[x_0, x_1] - f'(A)}{h/3}$$

(c) The $i=1$ instance of the equation for $\vec{c}$ in terms of $3f[x_{i-1}, x_i, x_{i+1}]$ plus the new equation gives

$$2c_0 + c_1 = \frac{f[x_0, x_1] - f'(A)}{h/3}$$

$$\frac{1}{2}c_0 + 2c_1 + \frac{1}{2}c_2 = 3f[x_0, x_1, x_2]$$

and the unchanged other equations:

$$\frac{1}{2}c_1 + 2c_2 + \frac{1}{2}c_3 = 3f[x_1, x_2, x_3]$$

$$\frac{1}{2}c_2 + 2c_3 + \frac{1}{2}c_4 = 3f[x_2, x_3, x_4]$$

$$\vdots \qquad \vdots$$

So instead of $N_{red,n-1}$ we get

$(2I+M)\vec{c}$ is given, where M is $n \times n$

for $\vec{c} = (c_0, c_1, \ldots, c_{n-1})$, where M is

$$M = \begin{bmatrix} 0 & 1 & & & & \\ 1/2 & 0 & 1/2 & & & \\ & 1/2 & 0 & 1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & 1/2 \\ & & & & \frac{1}{2} & 0 \end{bmatrix}.$$

Hence $\|M\|_\infty = 1$, since the new row has row sum 1, and the modified 2^{nd} row has row sum 1.

(4) The generalized mean-value theorem states that

$$f[x_0,\ldots,x_3] = \frac{f^{(3)}(\xi)}{3!}$$

for some ξ in any interval containing $x_0,\ldots,x_3$. Since $x_i = i/3$, this interval can be as small as $[0,1]$, i.e. $\xi \in [0,1]$.

Since $f(x) = \sin(x)$, $f'''(x) = -\cos(x)$.

Hence

$$|f[x_0,\ldots,x_3]| = \frac{|-\cos(\xi)|}{3!} \leq \frac{1}{3!} = \frac{1}{6}$$

(since we only know $\xi \in [0,1]$, $\cos(\xi)$ can be as large as 1).

(5) There error ir interpolation theorem says that

$$|f(x) - p_3(x)| \leq \left| \frac{f^{(4)}(\xi)}{4!} \right| \left| \prod_{i=0}^{3} (x - x_i) \right|$$

So $|f^{(4)}(\xi)| = |\sin(\xi)|$ which is bounded by 1 (or, more carefully, since x and $x_0, \ldots, x_3$ lie on $[0,1]$) $|\sin(\xi)| \leq \sin(1)$ (which is < 1 since $1 \leq \pi/2$)

Also: $\prod_{i=0}^{3}(x - x_i) = \left(\frac{1}{2} - 0\right)\left(\frac{1}{2} - \frac{1}{3}\right)\left(\frac{1}{2} - \frac{2}{3}\right)\left(\frac{1}{2} - 1\right)$

$$= \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{-1}{6} \cdot \frac{-1}{2} = \frac{1}{144}$$

So

$$\left| \frac{f^{(4)}(\xi)}{4!} \right| \left| \prod_{i=0}^{3}(x - x_i) \right| \leq \frac{\sin(1)}{24} \cdot \frac{1}{144}$$

or just $\frac{1}{24} \cdot \frac{1}{144}$

(16) (a) To compute the coefficients $c_0, \ldots, c_3$ with $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

will give a Vandermonde matrix with condition number at least order $(10^n)^3$ (See HW 6, Prob 5). For $n=6$, this number is larger than $2^{53} \approx 10^{16}$, so we can't expect any precision from this calculation.

By contrast, the $L_j(x)$ involve products of $\dfrac{x - x_i}{x_j - x_i}$. Since x, x_i, x_j are numbers 10^{-n} times $0, 1/3, 2/3, 1, 1/2$, these numbers are normal. Hence we expect x, x_i, x_j to have absolute error in precision roughly $10^{-n} \cdot 10^{-16}$. Since $x - x_i$, $x_j - x_i$ are all close to 10^{-n}, we expect relative error roughly 10^{-16}.

Hence we expect $L_j(x)$ to be
within $6 \cdot 10^{-16}$ of relative precision
(since each $L_j(x)$ is the product of
6 terms). So we expect Lagrange
interpolation to be much more precise.

[The only exception would be if
$p(10^{-n}(3/2))$ had some unexpected
cancellation ... To be 100% sure you'd
have to compare $p(10^{-n}(3/2))$ to
$\sin(i)$ for $i = 0, 1, 2, 3 \ldots$]

(See HW6, Prob 5)

6(b) No: Use $p_n(x)$ to denote $p(x)$
for various values of $n \in \mathbb{N}$. Then
for any $n, m \in \mathbb{N}$ we have

$$r(y) = p_n(10^{-n} y) - p_m(10^{-m} y)$$

has roots at $y = 0, 1/3, 2/3, 3/3$; since $r(y)$ is of degree ≤ 3, and $r(y)$ has at least 4 roots, $r(y) = 0$ (i.e. $r(y)$ is the zero polynomial).

Hence $p_n(10^{-n} y) = p_m(10^{-m} y)$ for all y, and hence $p_n(10^{-n}(3/2)) = p_m(10^{-m}(3/2))$.

Compare with HW 7, Problem 3, and Midterm, Question 4.

[Note: The midterm solutions show that there are a number of ways of solving this problem, such as with Lagrange Interpolation.]

(7) (a) The divided difference formula gives

$$q(x) = p(x) + (x-x_1)\cdots(x-x_n) f[x_1, \dots, x_{n+1}]$$

where $f(x) = \sin(1/x)$. The flop count is:

flops to compute $f[x_1, \dots, x_{n-1}]$: $O(n^2)$

" " " $q(y_i)$ for each i:

$O(n)$ to compute $p(y_i)$,

$O(n)$ " " $(y_i - x_1)\cdots(y - x_n)$

$O(1)$ to multiply $(y_i - x_1)\cdots(y - x_n)$ times $f[x_1, \dots, x_{n+1}]$

So total of $O(n)$ flops per each $q(y_i)$

" " " $O(\ell n)$ for all $q(y_i)$, $i = 1, \dots, \ell$.

Hence total flops $= O(n^2) + O(\ell n)$

7(b) Each $L_j(y_i)$ requires you to:

$$f(x_j) \prod_{k \neq j} \frac{1}{x_j - x_k} \quad \text{(these can be reused over all } j\text{)}$$

$$\prod_{k \neq j} (y_i - x_k) = \frac{\prod_{k=1}^{n+1} (y_i - x_k)}{y_i - x_j} \leftarrow \text{these can be reused over all } k$$

For a total of

$O(n^2)$ flops for $f(x_j) \prod_{k \neq j} \frac{1}{x_j - x_k}$

per each y_i $\begin{cases} O(n) \text{ flops for } L_1(y_i), \ldots, L_{n+1}(y_i) \\ O(n) \text{ flops for } \sum_j \left(f(x_j) \prod_{k \neq j} \frac{1}{x_j - x_k} \right) L_j(y_i) \end{cases}$

Total $O(n^2 + ln)$

Remarks: See [A&G], page 305, for a more complete discussion.

Note: (7a) can both be done with $O(\ln n + n)$ operations: it suffices to compute the c_n such that

$$q(x) = p(x) + c_n(x-x_1)\cdots(x-x_n).$$

You can find c_n as

$$c_n = \frac{q(x_{n+1}) - p(x_{n+1})}{(x_{n+1}-x_1)(x_{n+1}-x_2)\cdots(x_{n+1}-x_n)},$$

Since you want $q(x_{n+1}) = \frac{1}{n+1}$, and since you can compute $p(x_{n+1})$. Since you can compute $p(x_{n+1})$ in $O(n)$ flops, and $(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)$ as well, you can find c_n in $O(n)$ flops instead of $O(n^2)$ flops.

Similarly, from Lagrange interpolation, you have

$$q(x) = p(x) + \frac{(x-x_1)\cdots(x-x_n)}{(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)}\left(\frac{1}{n+1}\right),$$ which

also takes $O(n)$ flops.

(8) (a) Using $f(y) = ay$

$$y_{i+1} = y_i + h \frac{f(y_i) + f(y_{i+1})}{2}$$

becomes

$$y_{i+1} = y_i + h \frac{ay_i + ay_{i+1}}{2}$$

i.e.

$$y_{i+1}\left(1 - \frac{ah}{2}\right) = y_i\left(1 + \frac{ah}{2}\right)$$

So

$$y_{i+1} = y_i \frac{1 + ah/2}{1 - ah/2}$$

(b) If $a < 0$, then $1 - ah/2 > 1$

$$\frac{1 + ah/2}{1 - ah/2} < 1 \quad \text{iff}$$

$$1 + ah/2 < 1 - ah/2 \quad \text{iff} \quad ah < -ah$$

which is true, since $a<0$.

$$\frac{1+ah/2}{1-ah/2} > -1 \text{ iff}$$

$$1+ah/2 > -1+ah/2 \quad \text{iff } 1>-1$$

which is true. Hence

$$\frac{1+ah/2}{1-ah/2}$$

(c) It suffices to show that

$$\frac{1+ah/2}{1-ah/2} > 1$$

Since $ah<2$, $ah/2<1$ so $1-ah/2>0$. Hence we can multiply both sides by $1-ah/2$ and the inequality above is equivalent to

$$1 + ah/2 > 1 - ah/2$$

so

$$ah/2 > -ah/2$$

which holds since $a, h > 0$.

(d) If $ah > 2$ then $ah/2 > 1$

so $1 - ah/2 < 0$. Hence

$$\frac{1 + ah/2}{1 - ah/2} < 0$$

and so the y_i alternate in signs (since $y_0 = 1$, so $y_0 \neq 0$).

The true values of $y(0), y(h), y(2h), \ldots$ with $y_0 = 1$ are

$$y(ih) = e^{a(ih)} 1 = e^{iah}$$

which are always positive.

9 (a)

$$\frac{y(t+h)-y(t)}{h}$$

$$= \frac{\left(y(t)+hy'(t)+\frac{h^2}{2}y''(t)+O(h^3)\right)-y(t)}{h}$$

$$= \frac{hy'(t)+(h^2/2)\,y''(t)+O(h^3)}{h}$$

$$= y'(t)+\frac{h}{2}y''(t)+O(h^3).$$

Also

$$\frac{y'(t+h)+y'(t)}{2} = \frac{y'(t)+hy''(t)+O(h^2)+y'(t)}{2}$$

$$= y'(t) + \frac{h}{2} y''(t) + O(h^2).$$

Hence both sides $= y'(t) + \frac{h}{2} y''(t) + O(h^2)$.

(b) By (a),

$$\frac{y(t+h) - y(t)}{h} = \frac{y'(t+h) + y'(t)}{2} + O(h^2)$$

$$= \frac{f(y(t+h)) + f(y(t))}{2} + O(h^2)$$

So

$$y(t+h) - y(t) = \frac{h}{2}\left(f(y(t+h)) + f(y(t))\right) + O(h^3)$$

and adding $y(t)$ to both sides yields

$$y(t+h) = y(t) + \frac{h}{2}\left(f(y(t+h)) + f(y(t))\right) + O(h^3)$$

(c) Because of the $O(h^3)$ term

above, the method (1) is accurate

to second order.