THE UNIVERSITY OF BRITISH COLUMBIA
CPSC 303: MIDTERM EXAMINATION – March 11, 2020

Full Name: xLASTNAMEx   Exam ID: xFIRSTNAMEx

Signature: ______________________   UBC Student #: xDIGITSx x1x2x3x4x5x6x7x8x

Important notes about this examination

1. You have **50** minutes to complete this examination.
2. One two sided (8.5” x 11”) sheet of notes is allowed.
3. Good luck!

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. No questions will be answered in this exam. If you see text you feel is ambiguous, make a reasonable assumption, write it down, and proceed to answer the question.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   i. speaking or communicating with other examination candidates, unless otherwise authorized;
   ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
   iii. purposely viewing the written papers of other examination candidates;
   iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Please do not write in this space:

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0. Identification

Please make sure that the following is your 4- or 5-character ugrad email id:

`xFIRSTNAMEx`

Your answer to each problem should be written on its page; if needed, you can use the back side of the page as well.

1. (12 POINTS, 2 POINTS PER CORRECT ANSWER)

Circle either (a),(b),(c), or (d) next to the word **ANSWER** in the following questions.

(1) In double precision, a number that is roughly 2^{100} times larger than 2^{-1074} is stored with roughly
 (a) 100 bits of precision
 (b) 100 digits of precision
 (c) 2^{100} bits of precision
 (d) 53 bits of precision

ANSWER: (a) (b) (c) (d)

Solution: (d): Such a number would be a normal number, which is stored with 53 bits of prescision (or 52 bits, depending on if you count the first 1).

(2) In double precision, the expression 3^m is evaluated exactly for a postive integer m if and only if
 (a) $3^m \leq 2^{53} - 1$
 (b) $m \leq 53$
 (c) $3^m \leq 2^{1024} - 1.$
 (d) $3^m \leq 2^{1023}(2 - 2^{-52}).$

ANSWER: (a) (b) (c) (d)

Solution: (a): 3^m is an odd integer, and so its binary representation ends in a 1. Hence if it requires more that 53 bits to write it down exactly, it cannot be evaluated exactly (as a normal number). (Also 3^m is reported as +Inf roughly when $3^m > 2^{1024}$.)

(3) The ∞-condition number (i.e., condition number in the ∞-norm) of the matrix

$$\begin{bmatrix} 100 & 0 \\ 0 & -1 \end{bmatrix}$$

is
 (a) 100,
 (b) 1,
 (c) 1/100,
 (d) undefined.

ANSWER: (a) (b) (c) (d)

Solution: (a): the condition number for diagonal matrices is easy to compute and (even in the p-norm for any $p \geq 1$ or $p = \infty$) equals $\max(|d_1|, |d_2|)$ divided by $\min(|d_1|, |d_2|)$ which is 100 divided by 1.

(4) Consider the recurrence

$$x_{n+2} = (4/3)x_{n+1} - (1/3)x_n.$$

If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1/3$, for n large what will $|x_n|$ be reported as?
(a) Inf
(b) 0 or some periodically repeating sequence of subnormal numbers.
(c) Nan
(d) 10^m where m is roughly between -15 and -19.

ANSWER: (a) (b) (c) (d)

Solution: (d): The general solution to this recurrence is $C_1 + C_2(1/3)^n$. Since $x_1 = 1/3$ is written in base 2 in double precision, instead of the exact solution, namely $C_1 = 0$ and $C_2 = 1$, you will numerically observe $C_1 \neq 0$ and small (this experiment was done on the homework). For this reason we observed $|x_n|$ tending to $|C_1|$, which is of the order of magnitude in (d).

(5) Consider the recurrence

$$x_{n+2} = (9/8)x_{n+1} - (1/8)x_n.$$

If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1/8$, for n large what will $|x_n|$ be reported as?
(a) Inf
(b) 0 or some periodically repeating sequence of subnormal numbers.
(c) Nan
(d) 10^m where m is roughly between -15 and -19.

ANSWER: (a) (b) (c) (d)

Solution: (b): The general solution to this recurrence is $C_1 + C_2(1/8)^n$. Since double precision works in base 2, and the x_n are $1/8^n$ in exact arithmetic, double precision records this exact value for some n. For n sufficienty large x_n becomes a subnormal number, and eventually is either reported as 0 or will repeat periodically near the smallest, positive, subnormal number.

(6) There exist $a_2, a_1, a_0 \in \mathbb{R}$ such that

$$\max_{1\leq x\leq 2}\left|x^3 + a_2x^2 + a_1x + a_0\right|$$

is
(a) 1/32,
(b) 1/64,
(c) 1/128,
(d) none of the above.
[Hint: it may help to recall that $T_3(x) = 4x^3 - 3x$ and $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$. Also, writing $1 \leq x \leq 2$ is equivalent to writing $x \in [1, 2]$.]

ANSWER: (a) (b) (c) (d)

Solution: (a): $T_3(x)$ takes values in $[-1, 1]$ for $x \in [-1, 1]$; so $T_3(x)/4 = x^3 +$ lower order tersm takes values of absolute value at most $1/4$ for $x \in [-1, 1]$, and hene $T_3(2x - 1)/4$ does so as well for $x/2 \in [1, 2]$; we get a polynomial with leading term x^3 by dividing $T_3(2x - 1)/4$ by 8, which then takes absolute values of at most $1/32$. Since the Chebyshev polynomials are as best you can do in this regard, you cannot improve upon the $1/32$.

2. (6 POINTS)

Find $c_1, c_0 \in \mathbb{R}$ such that the polynomial $p(x) = x^2 + c_1 x + c_0$ satisfies

$$\max_{x \in [1,3]} |p(x)| = 1/2 \ .$$

[Writing $x \in [1,3]$ is the same as writing $1 \leq x \leq 3$.] **Show your work in finding c_1, c_0, but you do not need to justify why your method works.**

Solution:

Method 1: you want a parabola symmetric about $x = 2$ (since 2 is the midpoint or average of 1 and 3) that is moved down so that the endpoints $1, 3$ have the same absolute values as midpoint: so the parabola is $(x-2)^2$, which is 0 at $x = 2$ and 1 at $x = 1, 3$ moved down by $1/2$ to balance the 0 and 1 values: Hence we get

$$p(x) = (x-2)^2 - 1/2 = x^2 - 4x + 4 - 1/2 = x^2 - 4x + 7/2.$$

So $c_0 = 7/2$ and $c_1 = -4$.

Method 2: for the interval $[-1, 1]$, instead of $[1, 3]$ we know that one can similarly achieve $1/2$ as the lowest max, uniquely with $q(x) = x^2 - 1/2$. The map $x \mapsto x - 2$ translates $[1, 3]$ to $[-1, 1]$, so we have

$$p(x) = q(x-2) = (x-2)^2 - 1/2 = x^2 - 4x + 7/2.$$

So $c_0 = 7/2$ and $c_1 = -4$.

3. (12 POINTS)

Let $f(x) = x^2$. Consider the a unique polynomial $p_1(x) = c_0 + c_1 x$ such that $f(1) = p(1)$ and $f(3) = p(3)$.

(1) Find c_0, c_1 by any method you like; **show your work.**
(2) Use the "error in polynomial interpolation" formula find an upper bound on the error $|f(x) - p_1(x)|$ that is valid for all $x \in [1, 3]$; **show your work.**

Solution: [Remark: it was announced during the midterm that p and p_1 were meant to be the same. Below we will use p_1. If you took this exam in Access and Diversity, we will take a special look at your exam to see if this appeared to be problemic for you.]

(1) $c_0 + c_1 = f(1) = 1$ and $c_0 + 3c_1 = f(3) = 9$ yields, by subtracting the two equations. $2c_1 = 8$, so $c_1 = 4$, and hence $c_0 = -3$. Hence $p_1(x) = 4x - 3$.
(2) The "error in polynomial interpolation" formula implies that

$$\max_{x\in[1,3]} |f(x) - p(x)| \le \left(\max_{\xi\in[1,3]} |f''(\xi)| \right) \left(\max_{x\in[1,3]} \bigl|w(x)\bigr| \right),$$

where $w(x) = (x - 1)(x - 3)$. Since $f''(x) = 2$, we have

$$\left(\max_{\xi\in[1,3]} |f''(\xi)| \right) = 2.$$

To find the maximum of $|w(x)|$, where $w(x) = (x-1)(x-3)$, for $x \in [1, 3]$, it suffices to examine all the local maxima and minima of $w(x)$ on $[1, 3]$: at the endpoints $x = 1, 3$, the value of $w(x)$ is 0, and any other extrema are found by differentiating $w(x)$ and setting it to zero; since

$$w'(x) = (x^2 - 4x + 3)' = 2x - 4,$$

the only other local min/max is at $x = 2$, where $w(x) = -1$. Hence

$$\left(\max_{x\in[1,3]} |(x - 1)(x - 3)| \right) = 1,$$

and hence the error in polynomial interpolation formula shows that

$$\max_{x\in[1,3]} |f(x) - p_1(x)| \le 2 \cdot 1 = 2.$$