CPSC 303, March 13:

Dissipation of Energy in Splines:

$$\frac{h_{i-1}}{h_{i-1}+h_i} c_{i-1} + 2c_i + \frac{h_i}{h_{i-1}+h_i} c_{i+1} = \underbrace{3 f[x_{i-1},x_i,x_{i+1}]}_{\substack{\text{energy of } f \\ (\text{2nd difference, like } f''(\xi))}}$$

add

= 1

off diagonal stuff sums to 1

$$\begin{bmatrix} 2 & \cdot & & & & \\ \cdot & 2 & \cdot & & & \\ & \cdot & 2 & \cdot & & \\ & & \cdot & 2 & \cdot & \\ & & & \ddots & \ddots & \ddots \\ & & & & \cdot & 2 \end{bmatrix} \begin{array}{c} c_0 \\ \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \\ c_n \end{array} = \begin{bmatrix} \\ 3f[x_{i-1},x_i,x_{i+1}] \\ \\ \end{bmatrix}$$

energy term

nearest neighbour effect

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \underbrace{\text{remainder}}_{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \;\approx\; \frac{1}{n} + \text{order}\left(\frac{1}{n^2}\right)}$$

$$\underbrace{1 - \frac{1}{2}}_{\frac{1}{4}} + \underbrace{\frac{1}{4} - \frac{1}{8}}_{\frac{1}{8}} + \underbrace{\frac{1}{16} - \frac{1}{32}}_{\frac{1}{32}} + \cdots = \frac{2}{3}$$

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \cdots$$

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1$$

$$= 1 - x + x^2 - \cdots + (-x)^{n-1} + t_n$$

$$t_n = (-x)^n + (-x)^{n+1} + \cdots = (-x)^n \frac{1}{1+x}$$

decreases "geometrically" like $|x|^n$ ...

$$(I+A)^{-1} = I - A + A^2 - A^3 + A^4 + \cdots \quad \text{if } \|A\| < 1$$

$\|A\|_p$, $\|A\|_\infty$ ...

$$\cancel{(I+A)^{-1}} = \underbrace{I - A + A^2 - A^3 + \cdots + (-A)^{n-1}}_{} + \overbrace{T_n}^{\text{remainder}}$$

If $\|A\|_\infty < 1$, then converges,

and $\|T_n\|_\infty \le \|T\|_\infty^n \dfrac{1}{1-\|T\|_\infty} \leftarrow \left\{ (-x)^n \quad \dfrac{1}{1+x} \right.$

$$(I+A)^{-1} = I - A + A^2 - A^3 + \dots$$

remainder term decays like $\|A\|_\infty^n$

$$\left\| \frac{1}{2} \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} \right\|_\infty \le 1/2$$

$\leftarrow$ row sums in $| \, | \le 1/2$

Comfortable: $\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$,

$\|A+B\|_\infty \le \|A\|_\infty + \|B\|_\infty$

$\vdots$

$2I + B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\text{Energy}(f)}_{\text{constants}}$

off diagonal stuff has $\|B\|_\infty = 1$

's

$\|B/2\|_\infty \le \frac{1}{2}$

$I + B/2 = \frac{1}{2}$!

$\|B\|_\infty \le 1$

$\|B/2\|_\infty \le 1/2$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \left[I - \frac{B}{2} + \frac{B^2}{4} - \frac{B^3}{8} + \cdots \right] \text{Energy}(f)$$

Splines: $\min \int_{t_0 = A}^{t_n = B} |u''(x)|^2 \, dx$

s.t. $\boxed{u(t_0) = f(t_0),\ u(t_1) = f(t_1), \ldots,\ u(t_n) = f(t_n)}$

each piece $S_i(x) = a_i + (x - t_i) b_i + \underbrace{(x - t_i)^2 c_i}_{\text{enery}} + (x - t_i)^3 d_i$

$S_i(t_i) = u(t_i) = f(t_i)$

a_i given

$u''(t_0) = 0 = c_0$

$u''(t_n) = 0 = c_n$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$