CPSC 303.

My office hours tomorrow: 4-6 pm

Today's 2-3 pm class: "office hours"

(1) Chebyshev Polys: in $[a,b]$

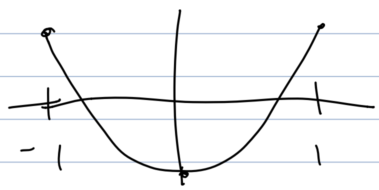
(2) Condition Nums & Relative Error:

Rel Error: $x, \hat{x}$ vs $\hat{x}, x$

Chebyshev Polys:

$\cos 2\vartheta = 2\cos^2\vartheta - 1$

on $[-1,1]$



$T_2(x) = 2x^2 - 1$

$= 2\left(x^2 - \frac{1}{2}\right)$

So want

$(x-x_0)(x-x_1)$ s.t. $x^2 +$ lower order in x

$$F(x_0,x_1) = \max_{-1 \le x \le 1} \left|(x-x_0)(x-x_1)\right|$$

$$\min_{x_0,x_1 \in \mathbb{R}} F(x_0,x_1)$$

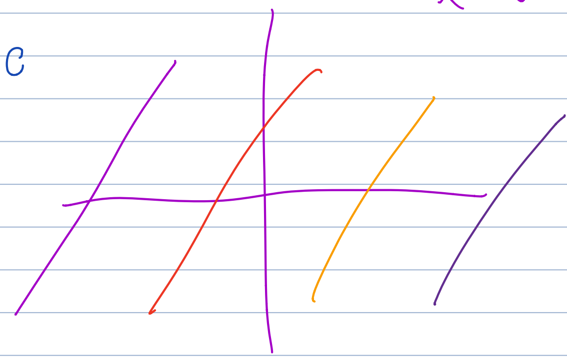
Thm: $\cos n\vartheta = 2^{n-1}\cos^n\vartheta + ?\cos^{n-2}\vartheta + ? \ldots$

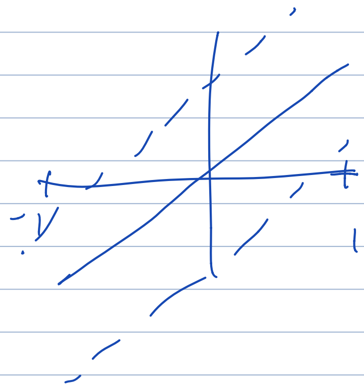
$T_n(x) = \underline{2^{n-1}}\, x^n + \underline{\quad}\, x^{n-2} + \underline{\quad} \ldots$

$x - x_0$

$$T_1(\cos(1\cdot\vartheta)) = \cos\vartheta + c$$

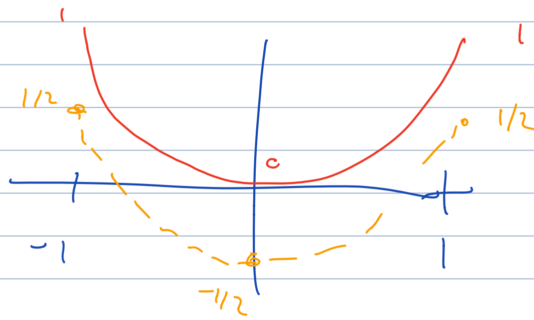
$$T_1(x) = x + c$$





$=$

$$(x - x_0)(x - x_1)$$

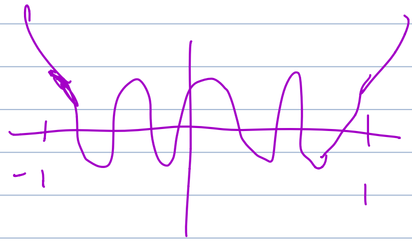


$$= \underbrace{x^2 + c_1 x + c_0}$$

T_4

$$T_n(x) = 2^{n-1}x^n + c_{n-2}x^{n-2} + c_{n-4}x^{n-4} \dots$$

$$= 2^{n-1}\underbrace{\left(x^n + \frac{c_{n-2}}{2^{n-1}}x^{n-2} + \frac{c_{n-4}}{2^{n-1}}x^{n-4} + \dots\right)}_{\text{between } [-1,1] \text{ for } |x| \le 1}$$



find some bound in $[a,b]$

$\downarrow$

ξ is in the smallest interval containing $x, x_0, \dots, x_n$

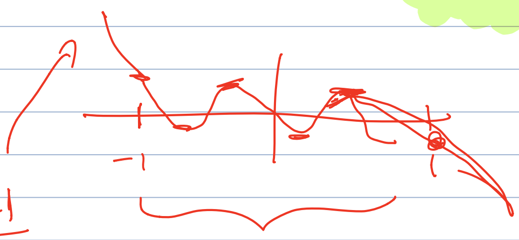
Error in interp:

$$f(x) - \underbrace{p_n(x)}_{\text{interp } f \text{ at } x_0,\dots,x_n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{\text{can control which } x_0,\dots,x_n \text{ in } [a,b]}$$

const dep on n

Chebyshev poly

describe them in $[-1,1]$



$\frac{T_n(x)}{2^{n-1}}$

length 2

-1 $\qquad$ 1

what if $[3,5]$?

adding 4

$g(x) = x+4$

$g^{-1}(y) = y-4$

$[-1,1] \leftarrow [3,5]$

$$\boxed{Q(x) = (x-x_0)\cdots(x-x_n)}$$

$[-1,1] \longleftarrow [3,5]$

$Q(y-4)$

(Now we are interested in $[3,5]$ rather than $[-1,1]$)

$Q(x) = x^n + \text{lower} \rightsquigarrow Q(y-4) = (y-4)^n + \text{lower order } (y-4)$

$y^n(-4y^{n-1} + \text{lower}) \quad -$

$x \mapsto x+4 = y$

$x-4 \leftarrow x$

$x = y-4$

$y-4 \leftarrow y$

WE WANT

Optimal poly on $[-1,1]$

(1) In terms of something about the values of poly

(2) It must begin x^n

poly $p(x)$

$p(-1) =$ blah

$p(-0.5) =$ somethin

$p(1) = \ldots$

Optimal poly on $[3,5]$

(1) In terms of something about the values of poly

(2) It must begin x^n

$p(x-4)$

$p(3-4) = p(-1) \ldots$

$\vdots$

$p(5-4) = p(1) \ldots$

$p(x) = (x-x_0)\ldots(x-x_n)$

roots

$p(x-4)$

$= (x-4-x_0)(x-4-x_1)\ldots(x-4-x_n)$

roots $4+x_0, \quad 4+x_1, \ldots$

$x_0 \quad 4+x_0$

$x_1 \quad 4+x_1$

$\ldots$

(2) Condition Nums & Relative Error:

Rel Error: $x, \hat{x}$ vs $\hat{x}, x$

Rel Error in p-norm $(\hat{x}, x)$ $\underset{\rightarrow}{\underline{\text{def}}}$ $\dfrac{\|\hat{x}-x\|_p}{\|x\|_p}$ $\longleftrightarrow$ $(x, \hat{x})$ $\dfrac{\|x-\hat{x}\|_p}{\|\hat{x}\|_p}$

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rel error $(1.1, 1.0) = \dfrac{|1.1-1.0|}{|1.0|} = \dfrac{|0.1|}{|1.0|} = \dfrac{1}{10}$

same

same

rel error $(1.0, 1.1) = \dfrac{|1.0-1.1|}{|1.1|} = \dfrac{|0.1|}{|1.1|} = \dfrac{1}{11}$

bigger? why?

$$|1.1| \leq |1.0| + |0.1|$$

triangle ineq

$$\|\hat{x}\|_p \leq \|x\|_p + \|x-\hat{x}\|_p$$

Say $\dfrac{\|\hat{x}-x\|_p}{\|x\|_p} = \dfrac{1}{10} \iff \boxed{\|\hat{x}-x\|_p = \frac{1}{10}\|x\|_p}$

$$\|x\| - \|\hat{x}-x\| \leq \|\hat{x}\| \leq \|x\| + \|\hat{x}-x\|$$

$$\|x\|\left(1-\frac{1}{10}\right) \leq \|\hat{x}\| \leq \|x\|\left(1+\frac{1}{10}\right)$$

$=$

(1) $\mathrm{cond}_p(A) = \|A\|_p \, \|A^{-1}\|_p$

(2) $\mathrm{cond}_p(A) = \begin{pmatrix}\text{loss of rel error} \\ \text{in solving } Ax=b\end{pmatrix}$

$$= \max_{b,\hat{b}} \frac{\text{rel err}(A^{-1}\hat{b}, A^{-1}b)}{\text{rel err}(\hat{b}, b)}$$

when are small

say $= 10^7/3$

$$\max_{b,\hat{b}} \frac{\text{rel err}(A^{-1}\hat{\hat{\hat{b}}}, A^{-1}\tilde{b})}{\text{rel err}(\hat{\hat{\hat{b}}}, \tilde{b})}$$