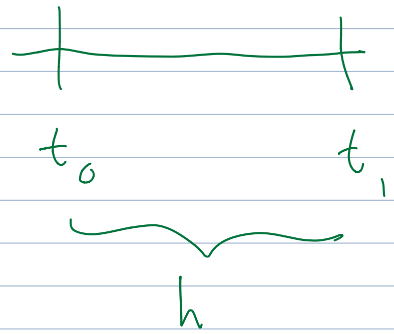
CPSC 303: Plan for 50% today, 50% Monday:

11.2 Error in Hermite interpolation between



is $\leq \dfrac{M_4}{384} h^4$

386?
384?
388?

See below: 384

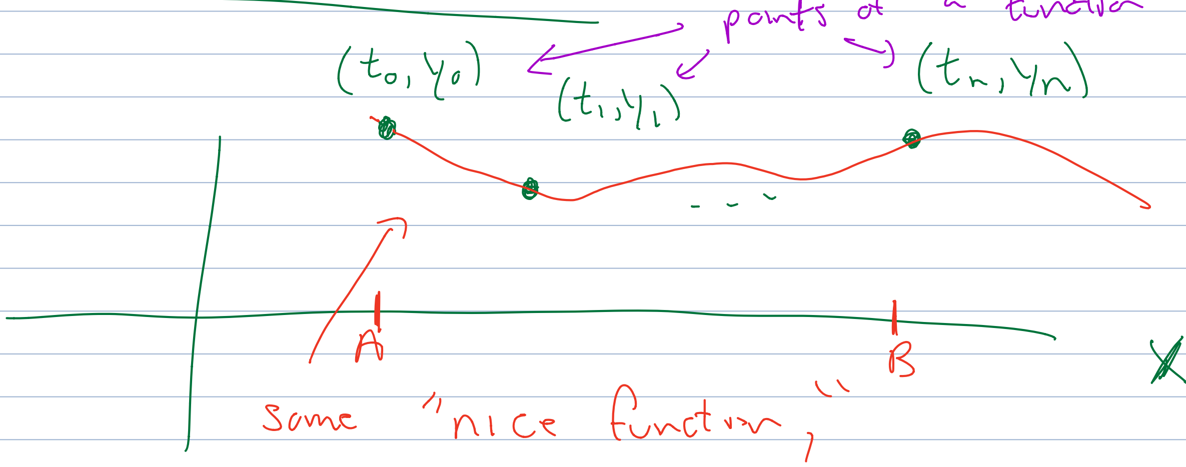
11.3 - Initial Value Problem

- Energy and $c_0, \ldots, c_m$
- Other energies, e.g. Dirichlet integral, length

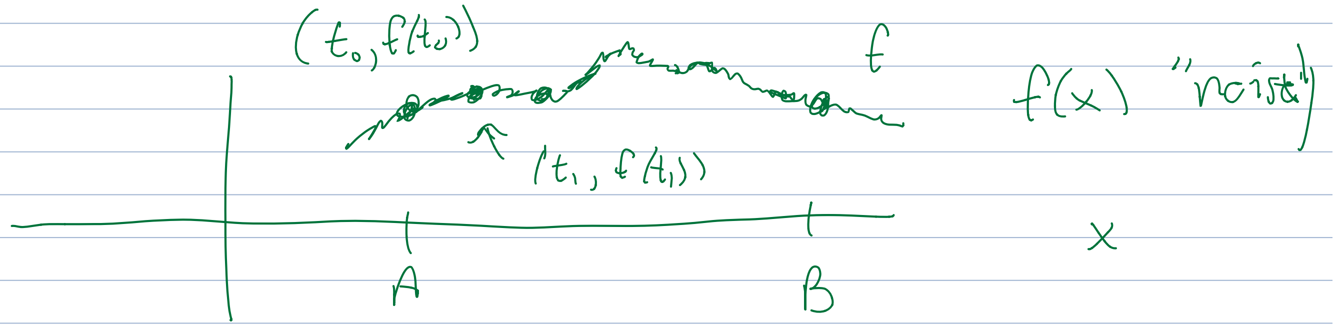
11.1 Promotes Piece-wise poly interpolation

Model for $f : [A,B] \to \mathbb{R}$

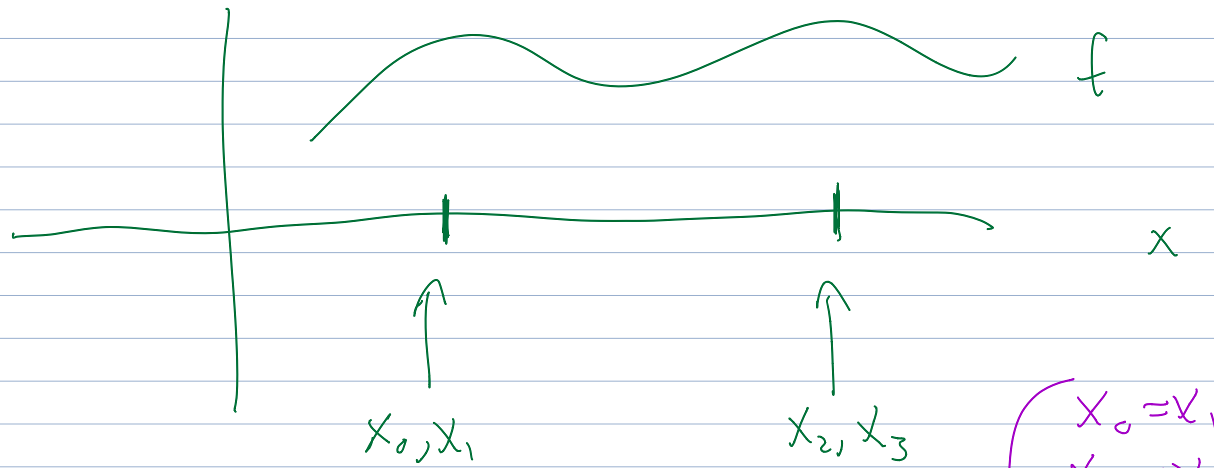
Spline:



some "nice function", $v(x)$, thru $(t_0,y_0), \ldots, (t_n,y_n)$



Practice Midterm Question: Interpolation



$x_0 = x_1$
$x_2 = x_3$

What is the error in interpolation if:

Fit $p_3(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) \cdot f[x_0, x_1, x_2]$

1 $+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$

2 Assume $|f''''(\xi)| \leq M_4$

$\left[M_4 := \max_{\xi \text{ in interval}} |f''''(\xi)| \right]$

Ans: $\dfrac{M_4}{4!}\max_{x_0 \le x \le x_3} \left| (x-x_0)(x-x_1)(x-x_2)(x-x_3) \right|$

der bound

factorial

$$\max_{x_0 \le x \le x_3} |v(x)|, \quad v(x) = (x-x_0)^2 (x-x_3)^2$$

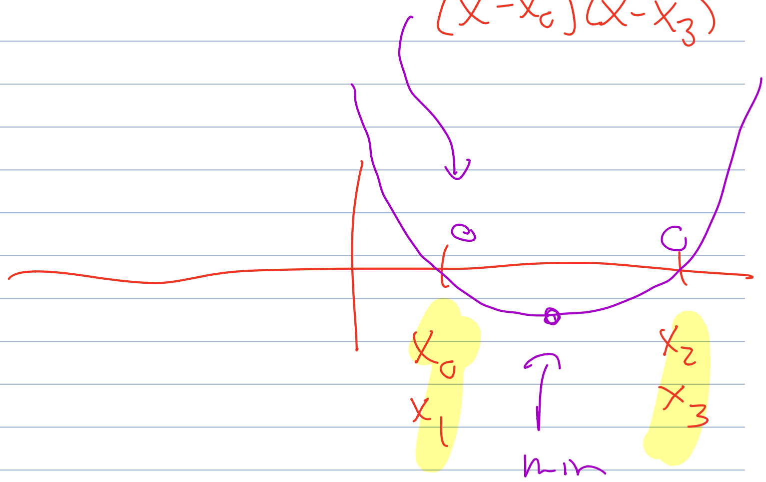
$x_0 = x_1, \; x_2 = x_3$

$v(x_0)$

$v(x_3)$

$v'(x) = 0$ ...

$$v(x) = \Big(\underbrace{(x-x_0)(x-x_3)} \Big)^2$$



min $(x-x_0)(x-x_3)$ is $x = \dfrac{x_0+x_3}{2}$

h

$h = x_3 - x_0$

$\max\Big(\qquad \Big)^2$ is

$$\left(\left(\frac{x_0+x_3}{2} - x_0\right)\left(\frac{x_0+x_3}{2} - x_3\right) \right)^2$$

$$\left(\frac{h}{2}\cdot\frac{(-h)}{2}\right)^2$$

$$\max_{x_0 \le x \le x_3} |v(x)| = \left(\frac{h^2}{4}\right)^2 = \frac{h^4}{16}$$

$$= \frac{(x_3 - x_0)^4}{16}$$

Ans: $\dfrac{M_4}{4!}\,\dfrac{h^4}{16}$ or $\dfrac{M_4}{4!}\,\dfrac{(x_3-x_0)^4}{16}$

Homework: $x_0 = 0$ $x_1 = 1$, specific.

$$4! = 24$$

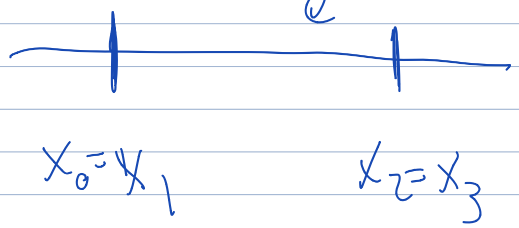
$$\begin{array}{r} 24 \\ 16 \\ \hline 144 \\ 24 \\ \hline 384 \end{array}$$

$$\frac{M_4}{384}\,h^4$$

Coefficients $f(x_0)$ $f(x_1)$ $f(x_2)$ $f(x_3)$

$f[x_1, x_0]$ $f[x_2, x_1]$ $f[x_3, x_2]$

if $x_1 = x_0$ $\quad f'(x_0) \quad \dfrac{f(x_2)-f(x_1)}{x_2-x_1} \quad f'(x_3) \quad x_2 = x_3$



Practice Midterm:

Generalized Mean Value
Divided Diff Derivative

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some ξ on smallest interval containing $x_0, \ldots, x_n$

(ξ depends on f and $x_0, \ldots, x_n$)

$f[x_0, x_1]$ 1st order
$f[x_0, x_1, x_2]$ 2nd order
$f[x_0] = f(x_0)$ 0th order

If
$\min \le f^{(n)}(\xi) \le \max$

then

$$\frac{\min}{n!} \leq f[x_0, \ldots, x_n] \leq \frac{\max}{n!}$$

$$\left| f[x_0, \ldots, x_n] \right| \leq \frac{\text{larger } |\max|, |\min|}{n!}$$

e.g. $f(x) = \sin x$

$$f^{(n)}(x) = \pm \begin{cases} \sin x \\ \cos x \end{cases}$$

$$-1 \leq f^{n}(\xi) \leq 1$$

$$\left| f^{n}(\xi) \right| \leq 1$$

e.g.

$$\max_{\xi \in [A,B]} \left| f^{(3)}(\xi) \right| \leq M_3$$

then

$$\left| f[x_0, x_1, x_2, x_3] \right| \leq \frac{M_3}{3!}$$

if $x_0, \ldots, x_3 \in [A, B]$

Remark: At start of Ch 16!

$x_0 < x_1 < x_2 < x_3$ in order, distinct

but

$$f[x_0, x_1, x_2, x_3] = f[x_1, x_0, x_3, x_2]$$

$$= f[\{x_0, x_1, x_2, x_3 \text{ in any order}\}]$$

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= f[x_0, x_1]$$

Error in interpolation

$$f(x) - \underbrace{p_n(x)}_{\text{deg } n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0) \cdots (x - x_n)$$

some ξ

"error" often mean

$$|f(x) - p_n(x)| \leq \text{some bound}$$

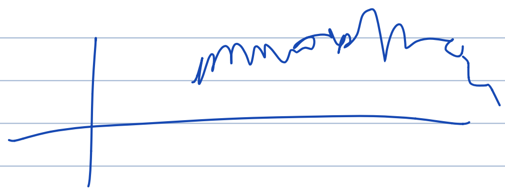
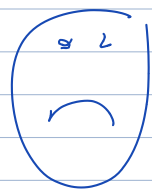
$f(x) = x^2$, $\quad x_0, x_1, x_2 \to p_2(x)$

$$|f(x) - p_2(x)| = 0 \qquad f'''(x) = 0$$

$f(x) = x^3 \quad x_0, x_1, x_2 \to p_2(x)$

$$f'''(x) = 6$$

$f(x) = \sin x$,  nice bounds

$f(x)$  

$$f(x) - p_2(x) = \frac{f'''(\xi)}{6}(x-x_0)(x-x_1)(x-x_2)$$

$$f(x) = x^3 \to \frac{6}{6}\underbrace{(x-x_0)(x-x_1)(x-x_2)}_{\text{true error}}$$