CPSC 303, March 2

Splines and ... condition number (!)

... divided differences (!)

Rem: Ch 11, Splines, really a continuation of Ch 10 in its ideas. [Algorithms are new.]

Good notation is <u>crucial</u>

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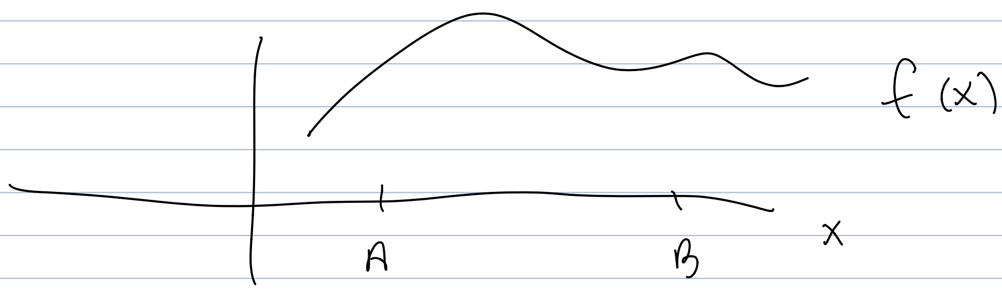
Fix

$A = a$ $\qquad$ $b = B$




You can measure values of $f(x)$

But maybe not values of $f'(x)$

(1) Pick $n$

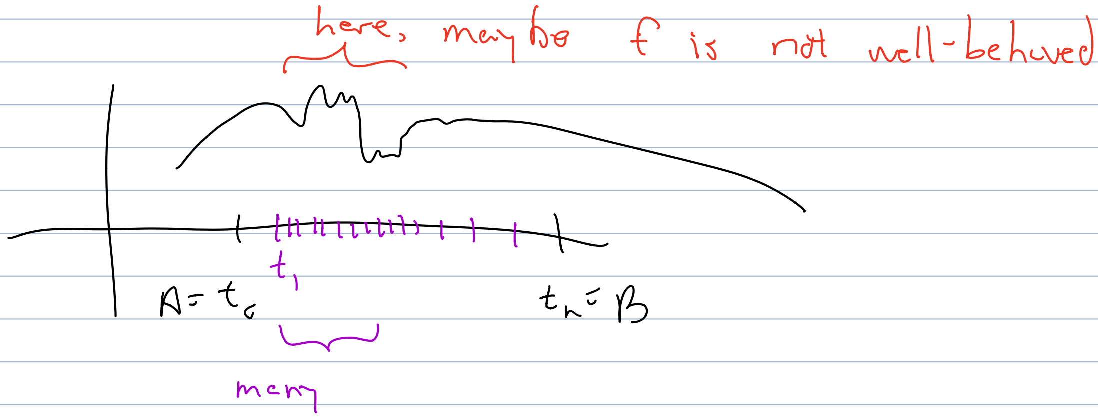
$h_0 = t_1 - t_0$ $\qquad$ $t_n - t_{n-1} = h_{n-1}$

$t_1$ $\quad$ $t_2$ $\quad$ ... $\quad$ $t_{n-1}$

(2) pick $\quad t_0 = A \quad$ $t_2 - t_1 = h_1$ $\quad$ ... $\quad$ $B = t_n$

$A = t_0 < t_1 \cdots < t_{n-1} < t_n = B$

So




$h_0 = t_1 - t_0,\ h_1 = t_2 - t_1,\ \ldots,\ h_{n-1} = t_n - t_{n-1}$ "small"

but not necessarily equal.

**(Cubic) Spline:** $v(x)$ which piecewise cubic polynomial,

i.e.

$$v(x) = \begin{cases} S_0(x) = a_0 + b_0(x-t_0) + c_0(x-t_0)^2 + d_0(x-t_0)^3, & t_0 \le x \le t_1 \\ S_1(x) = a_1 + b_1(x-t_1) + c_1(x-t_1)^2 + d_1(x-t_1)^3, & t_1 \le x \le t_2 \\ \vdots & \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x-t_{n-1}) + c_{n-1}(x-t_{n-1})^2 + d_{n-1}(x-t_{n-1})^3 & t_{n-1} \le x \le t_n \end{cases}$$

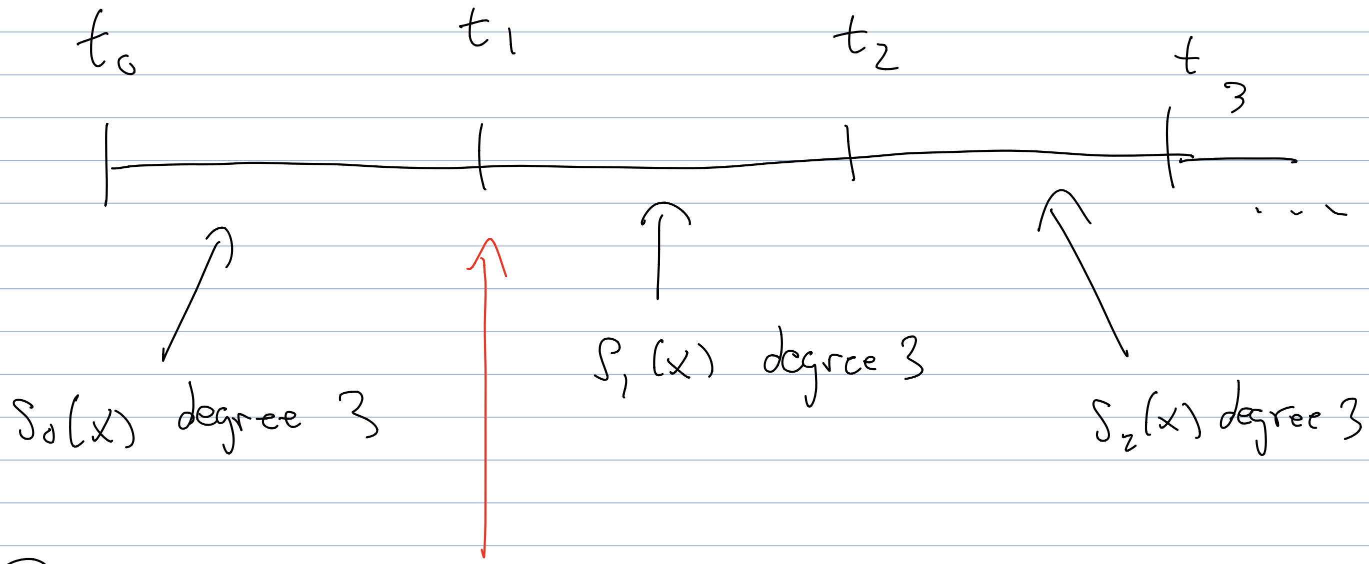
end points

$A = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = B$

break point

We require ① $v(x), v'(x), v''(x)$ to be continuous across break points

② $f : [A,B] \to \mathbb{R}$, $v(t_0) = f(t_0),\ v(t_1) = f(t_1), \ldots, v(t_n) = f(t_n)$




① above: $S_0(t_1)=S_1(t_1)$

$S_0'(t_1)=S_1'(t_1)$

$S_0''(t_1)=S_1''(t_1)$

② above: $S_0(t_0)$ = some given value, namely $f(t_0)$

$S_0(t_1)= S_1(t_1) = \dots \quad f(t_1)$

$S_1(t_2)= S_2(t_2) = \dots \quad f(t_2)$

small magic

better magic: $S_0(t_0)=S_0(x)\big|_{x=t_0}$

$$= \left(a_0+b_0(x-t_0)+c_0(x-t_0)^2+d_0(x-t_0)^3\right)\Big|_{x=t_0}$$

$f(t_0) = a_0$

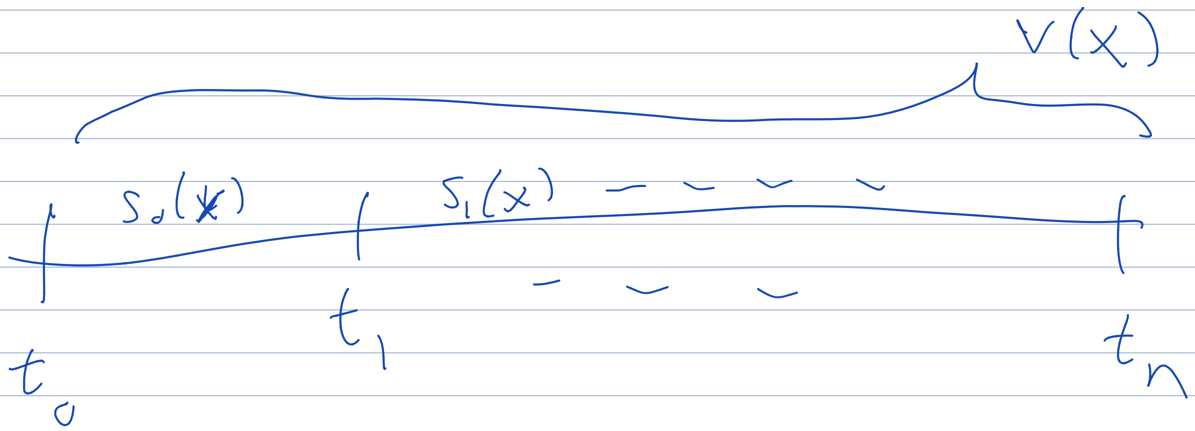
Similarly $a_1=f(t_1),\ a_2=f(t_2),\dots,\ a_{n-1}=f(t_{n-1})$

Last time, and today:

Parameters: $\{a_i, b_i, c_i, d_i \quad i = 0, \ldots, n-1\}$

$= 4n$ parameters

# conditions: $4n-2$ conditions




$$S_0(t_1) = S_1(t_1)$$
$$S_0'(t_1) = S_1'(t_1)$$
$$S_0''(t_1) = S_1''(t_1)$$

$\Big\}$ n-1 breakpoints

3 conditions

$v(t_0) = f(t_0)$, $v(t_1) = f(t_1)$, ... , $v(t_n) = f(t_n)$

$n+1$

Linear equations: $3(n-1) + (n+1) = 4n-2$

We specify two linear conditions,

we hope that there is a unique solution.

"Boundary conditions" "Initial Conditions"

⓪ Specify $v'(t_0)$, $v''(t_0)$ $t_0 = A$ is the left endpoint

① "Free" $v''(t_0) = v''(t_n) = 0$

② "Clamped" $v'(t_0)$, $v'(t_n)$ specified

③ Not-a-Knot $v'''$ is continuous across $t_1, t_{n-1}$

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$n = 1$ First

$n = 2$ Second

$$v(x) = S_0(x) = a_0 + b_0(x - t_0) + c_0(x - t_0)^2 + d_0(x - t_0)^3$$

$n = 1$

$A = t_0$ $t_1 = B$

⓪ $v(t_0) =$ given $= f(t_0)$

$v(t_1) =$ given $= f(t_1)$

$V'(t_0)$ = parameter , $V''(t_0)$ = parameter

$x$ = variable

$$V(\cancel{t}x) = a_0 + b_0(x - t_0) + c_0(x - t_0)^2 + d_0(x - t_0)^3$$

$$V(t_0) = a_0$$

$$V'(t_0) = \left( b_0 + 2c_0(x - t_0) + 3d_0(x - t_0)^2 \right)\Big|_{x = t_0}$$

$$= b_0$$

$$V''(t_0) = 2c_0$$

$$V'''(t_0) = 6d_0$$