CPSC: §10.7: (Limits) when $x_0, \dots, x_n$ not distinct

(1) Taylor series

(2) Hermite cubic interpolation ⟵

(3) Uniqueness and Multiple roots

(4) Remarks on Prob 1(c,d) and different implementations of MATLAB

mathworks.matlab.com

Taylor series: $x_0, x_1, \dots, x_n$

fix $t \in \mathbb{R}$, $\lim_{x_0, \dots, x_n \to t}$ (interpolation)

Look at

$p_n(x)$: interpolates f at $x_0, \dots, x_n$ (all distinct)

$$p_n(x) = f[x_0] + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2]$$

$$+ \dots + (x-x_0) \dots (x-x_{n-1}) f[x_0, x_1, \dots, x_n]$$

is there a limit?

$$\lim_{x_0, \ldots, x_n \to t} \quad \text{Taylor series}$$

$$f[x_0] \longrightarrow f(t)$$

$$f[x_0, x_1] = \text{slope secant line of } f \text{ at } x_0, x_1 = f'(\xi) \longrightarrow f'(t)$$

(assume f' exists and is continuous near t)

$$f[x_0, x_1, x_2] = f''(\tilde{\xi})$$

$\tilde{\xi}$ depends on f, x_0, x_1, x_2

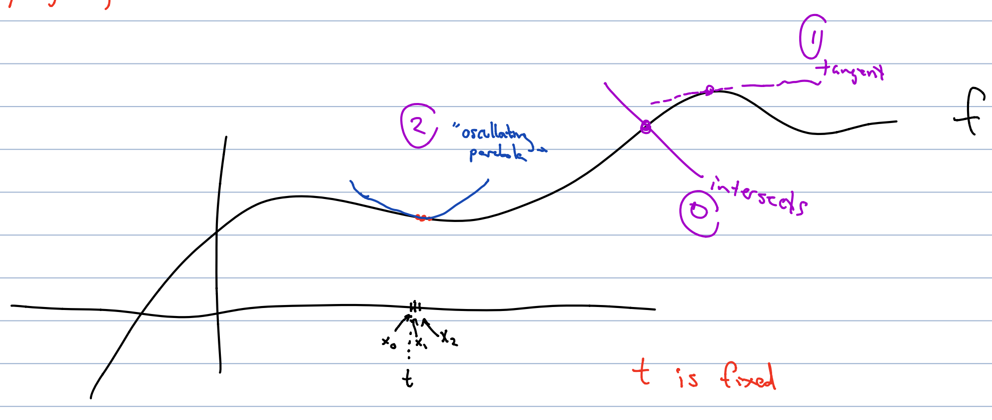
$$\lim_{x_0, x_1, x_2 \to t} f''(\hat{\xi}) = f''(t)$$

$(x_0, x_1, x_2) \in \mathbb{R}^3$

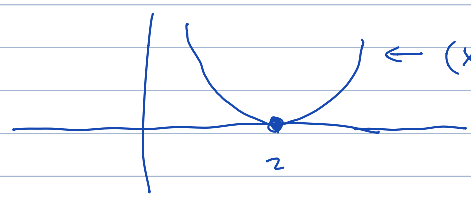
$\to (t, t, t)$

⋮

Example



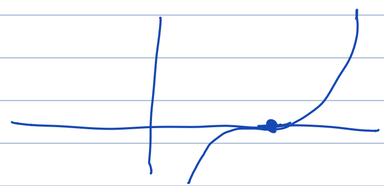
t is fixed



$\leftarrow (x-2)^2 = f(x)$

$f(2) = 0$

$f'(2) = 0$



$(x-2)^3$

$$\lim_{\substack{x_0,\dots,x_n \to t \\ t \text{ fixed}}} \left(f[x_0] + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + \dots + (x-x_0)\dots(x-x_{n-1}) f[x_0,x_1,\dots,x_n] \right)$$

($f[x_0] \to f(t)$, $f[x_0,x_1] \to f'(t)$, $f[x_0,x_1,x_2] \to f''(t)/2$)

$x-x_0 \to x-t$ $\quad (x-x_0)(x-x_1) \to (x-t)^2$

$$\lim = f(t) + (x-t)f'(t) + (x-t)^2 \frac{f''(t)}{2} + \dots + (x-t)^n \frac{f^{(n)}(t)}{n!}$$

(Taylor series)

How accurate?

Error in Poly Interp: $(x-x_0)\cdots(x-x_n)\dfrac{f^{(n+1)}(\xi)}{(n+1)!}$

$x_0,\dots,x_n \to t$? $\quad (x-t)^{n+1}\dfrac{f^{(n+1)}(\xi)}{(n+1)!}$ ← Remainder term in Taylor series

where ξ is in any interval containing x, t

Conventions: at first $x_0 < x_1 < \dots < x_n$

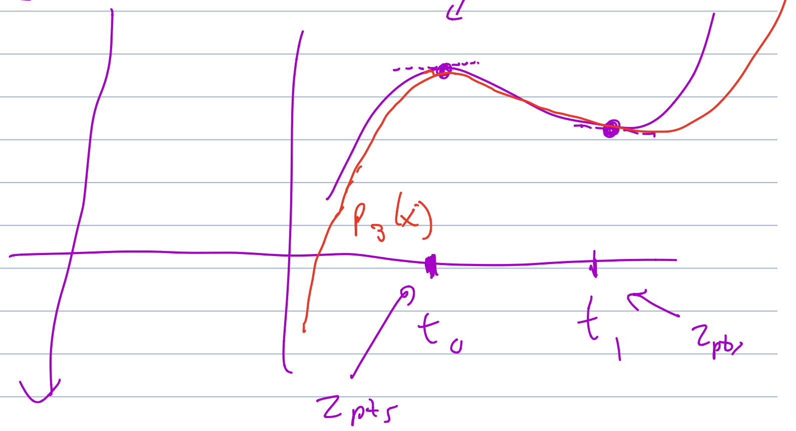
comes adding "x_{n+1}" ↔ x

Hermite cubic interpolation:

fit (value and its derivative)

$x_0 \quad x_1 \quad x_2 \quad x_3$ ← 4 pts

fix $\quad t_0 \quad t_1$



fits together in cubic

$$p_3(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$\lim_{\substack{x_0, x_1 \to t_0 \\ x_2, x_3 \to t_1}} \Big(= f[x_0] + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0,x_1,x_2,x_3] \Big)$$

Unique?

$$\to p'(x) = c_1 + 2c_2 x + 3c_3 x^2$$

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$p(2)$ given $\quad p(2) = c_0 + 2c_1 + 4c_2 + 8c_3$

$p'(2)$ given $\quad p'(2) = c_1 + 4c_2 + 12c_3$

$p(3)$ given $\quad p(3) = c_0 + 3c_1 + 9c_2 + 27c_3$

$p'(3)$ given $\quad p'(3) = c_1 + 6c_2 + 27c_3$

solve $c_0 + 2c_1 + 4c_2 + 8c_3 = 0$

$c_1 + 4c_2 + 12c_3 = 0$

$c_0 + 3c_1 + 9c_2 + 27c_3 = 0$

$c_1 + 6c_2 + 27c_3 = 0$

Homogeneous form

If $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$

s.t. $p(2)=0,\ p'(2)=0,\ p(3)=0,\ p'(3)=0$

can $p(x)$ be anything other than zero poly?

Method 1: If $p(x) \neq 0$, then $p(x) = (x-2)^2 (x-3)^2 r(x)$

impossible...

Method 2: Rolle's thm: $p'(2)=0,\ p'(3)=0,\ \left.\begin{matrix} p(2)=0 \\ p(3)=0 \end{matrix}\right\} p'(\xi)=0$

$p'(2)=0 \qquad p'(\xi)=0 \qquad p'(3)=0$

$p''(\xi_1)=0 \qquad p''(\xi_2)=0$

$p'''(\xi_3)=0 \Rightarrow c_3=0$

$p'''(x) = (c_3x^3 + c_2x^2 + \dots)''' = 6c_3 + 0$

$$p''(x) = (c_3 x^3 + c_2 x^2 + \dots)'' = 2c_2 \longrightarrow 0$$

$c_3 = 0$