CPSC 303

Today: § 10.5 & 10.6 slowly ...

Reminder: § 10.1 – 10.6 $x_0, \ldots, x_n$ are distinct

only § 10.7 " are not necessarily distinct

For us: looked at x_0, $x_1 = x_0 + \varepsilon$ $\varepsilon \to 0$

(Condition number, upper triangular systems : CPSC 302)

Elements of Ch 10...

Finish 10.4: Divided Difference Derivative Thm

Generalized Mean-Value Thm

Thm: f is n-times differentiable, $a, b \in \mathbb{R}$ s.t. $x_0, \ldots, x_n \in (a,b)$

then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \quad \xi \in (a,b)$$

Proof: (Magic) Apply Rolle's Thm n times to the right thing.

Thm: If f is $(n+1)$-times differentiable, if $x_0, \ldots, x_n$ distinct in (a,b)

and if $x \in (a,b)$, and if $p_n(x)$ is the unique poly of deg $\leq n$ s.t.

$p_n(x_i) = f(x_i)$, then

$$f(x) - p(x) = (x-x_0)\cdots(x-x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \text{s.t. } \xi \in (a,b)$$

(i.e. ξ depends on f, x, $x_0, \dots, x_n$) i.e. $\xi = \xi(f, x, x_0, \dots, x_n)$.

Corollary: If

$$\max_{\xi \in (a,b)} \left| f^{(n+1)}(\xi) \right| \leq M = M_{n+1, a, b, f}$$

then

$$\left| f(x) - p(x) \right| \leq \frac{M}{(n+1)!}$$

Proof: Let $x_{n+1} \in (a,b)$ and $p_{n+1}(x)$ be unique poly deg $\leq n+1$

s.t. $p_{n+1}(x) = f(x)$ for $x = x_0, x_1, \dots, x_{n+1}$

$$p_{n+1}(x) = p_n(x) + (x - x_0)(x - x_1) \cdots (x - x_n) f[x_0, x_1, \dots, x_{n+1}]$$

So

$$\underbrace{p_{n+1}(x_{n+1})} = p_n(x_{n+1}) + (x_{n+1} - x_0)(x_{n+1} - x_1) \cdots (x_{n+1} - x_n) f[x_0, \dots, x_{n+1}]$$

$$f(x_{n+1}) - p_n(x_{n+1}) = (x_{n+1} - x_0)(x_{n+1} - x_1) \cdots (x_{n+1} - x_n) \underbrace{f[x_0, \dots, x_{n+1}]}$$

Hence: nothing special about x_{n+1}:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$f(x) - p_n(x) = (x - x_0) \cdots (x - x_n) \cdot$$

if $x \neq x_0, \dots, x_n$. But also true if $x = x_i$, $i = 0, \dots, n$ since $f(x_i) = p_n(x_i)$

Real proof idea: We view $x = x_{n+1}$ as a new interpolation point.

Think of

$$f(x) - p_n(x) = \boxed{(x - x_0) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}}, \quad \text{as } x \in (a,b).$$

Application: Nice function $\sin(x)$, $\displaystyle \operatorname{err}(t) = \int_{-\infty}^{t} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$

So $\left|\sin^{(n+1)}(x)\right| = \dfrac{|\sin(x)|}{|\cos(x)|}$ or ≤ 1,

$(n+1)! = 1 \cdot 2 \cdots (n+1) =$ big pretty quickly

"Remainder Term"

$$(x-x_0)\cdots(x-x_n)\,\frac{\text{bound on } (n+1)\text{-st deriv}}{(n+1)!}$$

16.6: Fix $a=-1$, $b=1$, $x \in (-1,1)$, $x_0,\dots,x_n \in (-1,1)$

Say we can choose $x_0,\dots,x_n$ as we like. What is the best choice, i.e.

$$\max_{x\in(-1,1)} \left|(x-x_0)\cdots(x-x_n)\right|$$

as small as possible?

$n=1$:

x_0, x_1

$$\max(x_0,x_1) = \max_{x\in(-1,1)} \left|(x-x_0)(x-x_1)\right|$$

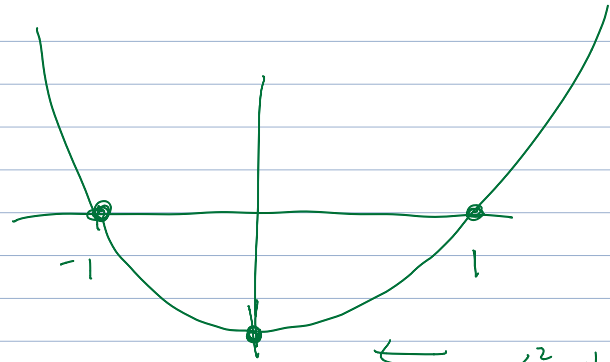
as small as possible...

Choose: $x_0 = x_1 = 0$: $(x-x_0)(x-x_1) = (x-0)(x-0) = x^2$

$$\max_{x\in(-1,1)} x^2 = 1.$$

Try $x_0 = 1$ $\quad x_1 = -1$:

$v(x) = (x-1)(x+1) = x^2 - 1$



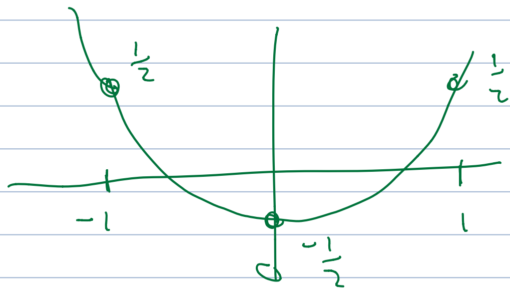
$\max\limits_{x \in (-1,1)} \left| x^2 - 1 \right| = 1$

$\leftarrow \quad x^2 - 1 = -1$

$(x - x_0)(x - x_1) = x^2 +$ lower order

$= \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right)$

$(x^2 - 0.5)$



turns out to be best

$$\cos(2\vartheta) = \cos^2\vartheta - \sin^2\vartheta = 2\cos^2\vartheta - 1$$

$$= 2\left(\cos^2\vartheta - 0.5\right)$$