CPSC 303:

Finish 10.4: { Divided Difference Derivative Theorem
"Generalized Mean-Value Theorem" ← me

10.5: Corollary of DDDT/GMVT:

{ Error in Polynomial Interpolation
"Generalized Remainder Theorem" ← me

10.6 Corollary of EPI/GRT:

How to choose $x_0, x_1, \ldots, x_n$ if you can

10.7 $f[x_0, \ldots, x_n]$ when $x_0, \ldots, x_n$ not necessarily distinct ...

Highlight ← BIG DEAL

We know:

Given $x_0 < x_1 < \cdots < x_n$ $n+1$ points in $\mathbb{R}$,

and $f: \mathbb{R} \to \mathbb{R}$,

(1) There is a unique poly

$$p_n(x) = c_0 + c_1 x + \cdots + c_n x^n \quad \text{s.t.} \quad p_n(x_i) = f(x_i), \quad i = 0, \ldots, n$$

(2) $c_n = f[x_0, \ldots, x_n]$

(3) $p_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$

$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0,\dots,x_n]$$

④ If $x_{n+1} > x_n > \dots > x_0$, x_{n+1} is new data point,

and $p_{n+1}(x)$ is unique deg $\le n+1$ polynomial with $p(x_i) = f(x_i)$
$i = 0, 1, \dots, n+1$

$$p_{n+1}(x) = f[x_0] + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2]$$

$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0,\dots,x_n]$$

$$+ (x-x_0)(x-x_1)\cdots(x-x_n)f[x_0,\dots,x_n,x_{n+1}]$$

$$= p_n(x) + (x-x_0)(x-x_1)\cdots(x-x_n)f[x_0,\dots,x_n,x_{n+1}]$$

Example: If $p(2) = f(2)$, $p(3) = f(3)$, $p(5) = f(5)$

with
$$p(x) = c_0 + c_1 x + c_2 x^2,$$

Then if you want $q(x) = \hat{c}_0 + \hat{c}_1 x + \hat{c}_2 x^2 + \hat{c}_3 x^3$

s.t. $q(x) = f(x)$ for $x = 2, 3, 5, 9$

$$q(x) = p(x) + (x-2)(x-3)(x-5)f[2,3,5,9] \Bigg\}$$

don't have to recompute much

Using 10.1:

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & 9 & 81 & \underset{729}{9^3} \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} = \begin{bmatrix} f(2) \\ f(3) \\ f(5) \\ f(9) \end{bmatrix}$$

"Newton polynomial" Divided Difference Interpolation Formula

" " Poly " "

Section 10.4

Langrange

$$, \ldots + y_3 \frac{(x-2)(x-3)(x-5)}{(9-2)(9-3)(9-5)}$$

Divided differences:

$$f[2] = f(2), \qquad f[2,3] = \frac{f(3)-f(2)}{3-2} \leftarrow 1^{st} \text{ div diff}$$

Mean-Value Thm: For any differentiable f,

$$f[2,3] = \frac{f(3)-f(2)}{3-2} = f'(\xi) \quad 1^{st} \text{ derivative}$$

Some $2 < \xi < 3$.

$f[2,3,5]$ or $f[x_0, x_1, x_2]$ 2nd div diff

There is a polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 = c_0 + c_1 x + f[x_0,x_1,x_2]\, x^2$$

s.t. $p(x) = f(x)$ at $x = x_0, x_1, x_2$

$e(x) = p(x) - f(x)$ "error in interpolation"

$e(x_0)=0 \qquad e(x_1)=0 \qquad e(x_2)=0$

$e'(\xi_1)=0 \qquad e'(\xi_2)=0$

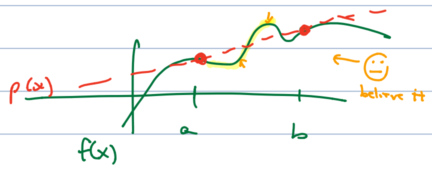
Using Rolle's thm $\quad e''(\xi) = 0$

$$e''(\xi) = p''(\xi) - f''(\xi) = 2f[x_0, x_1, x_2] - f''(\xi)$$

So $\quad f[x_0, x_1, x_2] = \dfrac{f''(\xi)}{2}, \quad x_0 < \xi < x_2$

Usual MVT : Have a, b, $a < b$,

$$p(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a} \quad \text{has} \quad p(x) = f(x),\ x = a, b$$



$p(x) - f(x)$ is zero at $x = a$, $x = b$

so $\underbrace{p'(\xi)}_{\frac{f(b)-f(a)}{b-a}} - f'(\xi) = 0$,

We set

$\text{error}(x) = p(x) - f(x)$

$\text{error}(a) = 0 = \text{error}(b)$

then $\quad \text{error}'(\xi) = 0$

Exact same method:

$p(x) = f(x)$ at $x = x_0, x_1, \ldots, x_{n+1}$

then $\quad e(x) = p(x) - f(x)$ is zero at $n+1$ points

so $\quad e'(x)$ " " " $\geq n$ "

$\quad e''(x)$ " " " $\geq n-1$ "

$\quad e^{(n)}(x)$ " " " ≥ 1 point

$$\Rightarrow n!\, f[x_0, \ldots, x_n] = f^{(n)}(\xi)$$

"Divided Diff Derivative Thm"

Generalized Mean-Value Thm

Cor! $e(x) = f(x) - p(x)$ $\qquad$ $p(x) = f(x)$ $\quad x = x_0, \ldots, x_n$

p degree $\leq n$

then

10.5

$$\boxed{e(x) = (x-x_0)\cdots(x-x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!}}$$

for some ξ in interval containing x and $x_0, x_1, \ldots, x_n$

looks like remainder in Taylor's thm

10.6

Cor: Say $x, x_0, \ldots, x_n \in (a,b)$ and say $M \in \mathbb{R}$

$|f^{(n+1)}(\xi)| \leq M$ for all $\xi \in (a,b)$

then $|e(x)| \leq |x-x_0|\,|x-x_1| \cdots |x-x_n| \dfrac{M}{(n+1)!}$