CPSC 303:

Divided differences: $\overbrace{f[x_0] = f(x_0)}^{0^{th}\text{ Div Diff}}$, $\overbrace{f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}^{1^{st}\text{ Div Diff}}$

Friday!

$$\left.f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}\right\} 2^{nd}\text{ Div Diff}$$

$$(1) \quad f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)}$$

compute

compute

$$+ \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

Rem: 3^{rd} Div Diff

$$f[x_0, \dots, x_3] = \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_0 - x_3}$$

$$= \frac{f[x_2, x_1, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \dots \frac{f(x_3)}{(x_3 - x_2)(x_3 - x_0)(x_3 - x_1)}$$

= (Lagrange formula) x^3-coeff in unique polynomial

$p(x) = C_0 + x C_1 + x^2 C_2 + x^3 C_3$ s.t. $p(x_i) = f(x_i)$, $i = 0, 1, 2, 3$

Divided diffs:

① $f[x_0, \dots, x_n] = f[x_0, \dots, x_n \text{ in any order}]$ "symmetric"

② Consequence of ⑤

③ "Mean-value thm" if f is n-times differentiable

$$f[x_0, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$ where

ξ is some value on the interval containing $x_0, \dots, x_n$

④ If f is n-times continuously differentiable

subtle

$f[x_0, \dots, x_n]$ is defined even if $x_0, \dots, x_n$ not necessarily distinct, and is continuous, etc.

⑤ If $x_0, \dots, x_n$ are any reals, not necessarily distinct, and p is the polynomial that agrees with f on $x_0, \dots, x_n$ and is of deg $\leq n$, then $p(x) = f[x_0] + (x-x_0) f[x_0, x_1]$

$$+ (x-x_0)(x-x_1) f[x_0, x_1, x_2] + \dots$$

Textbook

$$+ (x-x_0)(x-x_1) \dots (x-x_{n-1}) f[x_0, \dots, x_n]$$

$n=1$:

$$f[x_0,x_1] = \frac{f[x_0]-f(x_1)}{x_0-x_1}$$

Claim: If $x_0 < x_1$, and if given f function, $p(x) = \tilde{c}_0 + (x-x_0)\,\tilde{c}_1$ satisfies

$p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, then

$$\tilde{c}_0 = f(x_0), \quad \tilde{c}_1 = f[x_0,x_1]$$

$$\Longrightarrow \quad p(x) = f[x_0] + f[x_0,x_1]\,(x-x_0)$$

"secant line"

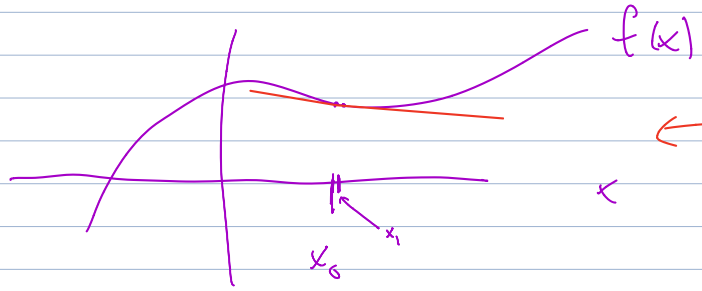
Pf: $p(x_0) = f(x_0)$

$$p(x_1) = f(x_0) + \frac{f(x_0)-f(x_1)}{x_0-x_1}(x_1-x_0)$$
$$= f(x_1)$$

Mean Value Thm: $\dfrac{f(x_0)-f(x_1)}{x_0-x_1} = \dfrac{f(x_1)-f(x_0)}{x_1-x_0} = f'(\xi)$

some $x_0 < \xi < x_1$

What if $x_1 \to x_0$



slope $\doteq f'(\xi)$

$\to f'(x_0)$

"$x_0 = x_1$" $p(x) = f(x_0) + (x - x_0)\, f'(x_0)$

$n = 2$:

We know $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, $p(x_2) = f(x_2)$

then

$$p(x) = c_0 + c_1 x + c_2 x^2,$$

we know $c_2 = f[x_0, x_1, x_2]$

<u>Claim</u>:

$$p(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

Why? (Trick):

$$q(x) = f(x_0) + (x - x_0) f[x_0, x_1] \qquad \begin{cases} q(x_0) = f(x_0) \\ q(x_1) = f(x_1) \end{cases}$$

Let's look for

$$= f(x_0) + (x - x_0) f[x_0, x_1] + (??) \underbrace{(x - x_0)(x - x_1)}_{=0 \text{ for } x = x_0, x_1}$$

(We choose (??) use Lagrange interp)

Aside: If $p(x) = f(x)$ for $x_0, \ldots, x_{n-1}$ and want

$\tilde{p}(x) = f(x)$ for $x = x_n$ as well, and $\tilde{p}(x)$ deg $\leq n$

$$\tilde{p}(x) = p(x) + \underbrace{??}_{\text{has to be } x^n \text{ coefficient, } c_n} (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Lagrange, Newton Dn Diff