CPSC 303

— Remarks on why MATLAB has (as in Homework 1,2):

largest positive number $\approx 2^{1024} \approx 1.8 \times 10^{308}$

smallest " " $\approx 2^{-1074} \approx 4.9 \times 10^{-324}$

MATLAB's realmin $\approx 2^{-1022}$, 2^{-1023} $\quad 2.2 \times 10^{-308}$

— More on divided differences

Main Thm: $f[x_0,\ldots,x_n] = \frac{1}{n!} f^{(n)}(\xi)$, $\xi \in (x_0, x_n)$

Pf: ... accidentally came across interpolation ...

Handout (Note) About Normal vs Subnormal in Double Precision

[Not covered in Ch 2 of [A & G]]

$$\boxed{\begin{array}{l}
\text{Idea:} \\
\text{"Normal" number} \quad \pm \, 1.b_1 b_2 \ldots b_{52} \times 2^m \\
\qquad\qquad\qquad\quad s \qquad b_1,\ldots,b_{52} \in \{0,1\} \qquad m \text{ integer,} \\
\text{base 2 arithmetic} \qquad\qquad\qquad\qquad -1022 \le m \le 1023
\end{array}}$$

Normal Number $\quad \underbrace{s}_{1}, \underbrace{b_1,\ldots,b_{52}}_{+\,52}, \quad + \underbrace{11 \text{ bits (binary digits) for } m}_{11} = 64 \text{ bits}$

Intuitively think of

base 10 sci-notation 6.27195×10^{2020}

$$m = \underbrace{-1022, \ldots, -1, 0, 1, \ldots, 1023}_{2046 \text{ values}}$$

(11 bits is 2048 possibilities)

$\boxed{2 \text{ leftover}}$

2 special values of the 11-bit "m" :

① Inf, -Inf, NaN, ...

② Revert to "subnormal form"

Form:

$$\underset{s}{\pm} \quad \underbrace{0.b_1 b_2 \ldots b_{52}}_{52 \text{ bits}} \times 2^{\overset{}{-1022}}$$

s , 52 bits , fixed

e.g. $\pm \; 0.\underline{000}\underbrace{10101110\ldots0}_{< 52 \text{ bits}} \times 2^{-1022}$

drop down to small power of 2

base 10 : $0.d_1 d_2 d_3 d_4 d_5$

$$\begin{cases} 0.12345 \\ 0.27193 \\ 0.\underline{000}21 \end{cases}$$

Variant $\pm\; 0.0000 b_1 \ldots b_{48} \times 2^{-1022 - \underbrace{b_{49} \ldots b_{52}}}$

Largest number in double precision:

$$+ \quad 1.b_1 \ldots b_{52} \quad \times 2^{1023}$$

$$\downarrow$$

$$(1.11111\ldots)$$

$$\left(2 - \frac{1}{2^{52}}\right)$$

Smallest pos number

$$+ \quad 0.\underbrace{000 \ldots 01}_{2^{-52}} \quad 2^{-1022}$$

$$2^{-52} \cdot 2^{-1022} = 2^{-1074}$$

HW 1, 2

$$X_{n+2} = (1+r)\, X_{n+1} - r X_n$$

general sol is

$$C_1 (1^n) + C_2 (r^n)$$

$r = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$\begin{bmatrix} X_{n+2} \\ X_{n+1} \end{bmatrix} = \begin{bmatrix} 1+r & -r \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix}$$

$x_0 = 1, \quad x_1 = r$

$$\begin{bmatrix} r^{n+1} \\ r^n \end{bmatrix} = \begin{bmatrix} 1+r & -r \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} r \\ 1 \end{bmatrix}$$

$r = \frac{1}{\text{power of 2}}$  , $\quad r = \frac{1}{3}, \frac{1}{5}, \frac{1-\sqrt{5}}{3}, \frac{1}{\sqrt{7}}$

$x_0 = 1, \quad x_1 = r$ $\left\{ \begin{matrix} \text{theoretically} \\ \text{exact arithmetic} \end{matrix} \right\}$ $x_n = r^n$

numerical : $x_n = C_1 + C_2 r^n$

C_1 small, but not 0

$C_2 = 1$

Another 3-term:

$$\left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0$$

general solution

$$x_n = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n$$

recurr

$$r^2 - \frac{5}{6} r + \frac{1}{6} = 0 \qquad x_{n+2} - \frac{5}{6} x_{n+1} + \frac{1}{6} x_n = 0$$

What about

$$x_{n+2} - 2x_{n+1} + x_n = 0 \quad ??$$

our method:

$$r^2 - 2r + 1 = 0$$

$$\boxed{(r-1)^2 = 0}, \qquad r = 1, 1$$

general solution: $C_1 1^n + C_2 1^n$??

$$x_n = \begin{cases} (C_1 + C_2) \cdot 1 \\ C \cdot 1 \end{cases}$$

$$x_n = \underbrace{C_1 + C_2 n}_{\text{linear polynomial}}$$

Recall:

..., 3 3 3 3 3 ...

1st Diff ... 0 0 0 0 ...

..., 4, 8, 12, 16, 20 ...

1st Diff ... 4 4 4 4 ...

2nd Diff ... 0 0 0 0 ...

And $(r-1)^3 = 0$

$$x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n = 0$$

Gen sol. $C_1 + C_2 n + C_3 n^2$

Differences, special case Divided Diffs

Added after class on Whiteboard:

$x_{n+1} - x_n = 0$ or $x_{n+1} = x_n$ gen sol. $x_n = C_1$

$x_{n+2} - 2x_{n+1} + x_n = 0$ gen sol $x_n = C_1 + C_2 n$

$= C_1(1^n) + C_2(1^n n)$

Div Diff:

$\ldots x_1, x_2, x_3, x_4, \ldots$

1st Diff $x_2 - x_1,\ x_3 - x_2,\ x_4 - x_3$

2nd Diff

$\ldots x_3 - 2x_2 + x_1,\ x_4 - 2x_3 + x_2, \ldots$

$x_{n+2} - 2x_{n+1} + x_n$