CPSC 303

- Differences
- Divided Differences ← Main technical tool in Ch. 16

Where do:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

come from ?

=

Differences!

"Difference Op"

$(Df)(x)$
$= f(x+1) - f(x)$

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9
$(1^{st}\ Diff)(f)$	-5	-3	-1	1	4-1=3	5	

	9	4	1	0	1	4	9
$(1^{st}\ Diff)(f)(x)$	-5	-3	-1	1	3	5	← $(Df)(x)$
$((2^{nd}\ Diff)(f))(x)$	2	2	2	2	2	...	
$(3^{rd}\ Diff)(f)(x)$	... =	0	0	0	0	0 ...	$(D^3 f)(x)$

$$y_{-2} \quad y_{-1} \quad\quad y_0 \quad\quad y_1 \quad y_2 \quad y_3$$

$$4 \quad\quad 1 \quad\quad 0 \quad\quad 1 \quad\quad 4 \quad\quad 9 \,\cdots$$

$$-3 \quad\quad -1 \quad\quad 1 \quad\quad 3 \quad\quad 5$$

$\uparrow \quad\quad \uparrow \quad\quad \uparrow \quad\quad \uparrow$

1ˢᵗ DIFF $\quad\quad y_{-1}-y_0 \quad\quad y_1-y_0 \quad y_2-y_1 \quad y_3-y_2$

2ⁿᵈ DIFF

$$(y_1 - y_0) - (y_0 - y_{-1})$$

$$\boxed{(y_1 - 2y_0 + y_{-1})} \quad \text{next} \quad (y_2 - 2y_1 + y_0)$$

3ʳᵈ DIFF

$$y_2 - 3y_1 + 3y_0 - y_{-1}$$

$$\left(3^{rd}\ \text{Diff}\right)\left(x \longmapsto x^2\right) = 0$$

$$\text{''} \quad (n \longmapsto n^2) = 0 \implies y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 0$$

$$y_{n+1} - y_n$$

$$(n+1)^2 - n^2 \;=\; (n^2 + 2n + 1) - n^2 \;=\; 2n + 1$$

$$(\text{Diff})(n \longmapsto n^2) \;=\; 2n + \text{lower order}$$

$$\text{''} \quad (n \longmapsto n^3) \;=\; (n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$= 3n^2 + \text{lower}$$

$$\sim \sim \quad 9 \quad 4 \quad 1 \quad 0 \quad 1 \quad 4 \quad 9 \quad \sim \sim$$

1st Diff $\quad \sim \sim -5 \quad -3 \quad -1 \quad 1 \quad 3 \quad 5 \quad 7 \quad \sim \sim \quad \big\}$ 2n + lower

2nd Diff $\quad \sim \sim \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad \sim \sim \quad \big\}$ 2

$$(\text{Diff})(n^2) = 2n + \text{lower}$$

$$(\text{Diff})^2(n^2) = (\text{Diff})(2n + \underbrace{\text{lower}}_{\text{const}}) = 2$$

$$\sim \sim \sim -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \sim \sim$$

3n+5 $\quad \sim \sim -1 \quad 2 \quad 5 \quad 8 \quad 11 \quad 14 \quad 17 \quad \sim \sim$

Diff $\quad \sim \sim 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad \sim \sim$

Diff^2 $\quad \sim \sim \sim 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \sim \sim$

$$\text{Diff}(y_n) = y_{n+1} - y_n$$

$$\text{Diff}^2(y_n) = \text{Diff}(y_{n+1}) - \text{Diff}(y_n)$$

$$= (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)$$

$$= y_{n+2} - 2y_{n+1} + y_n$$

if Diff^2 = 0, then $\quad y_{n+2} - 2y_{n+1} + y_n = 0 \quad \left(\begin{array}{c}\text{3 term} \\ \text{recurrence}\end{array}\right)$
for all n $\qquad\qquad\qquad$ for all n

$\Big\{$ "Differences" $\quad$ We need "Divided Differences" ...

Question: x :   -5   -2   0   1   3   4   9   10

$f(x) = x^2$:   25   4   0   1   9   16   81   100

Diff: (☹)   -19   -4   1   8   7   65   19

Can we correct "differences" for points that are not equally spaced ...

Answer! $f(x_1) - f(x_0)$ ⟶ $$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

x :   -5   -2   0   1   3   4   9   10

$f(x) = 3x + 2$: -13   -4   2   5   11   14   29   32

Diff: 9   6   3   6   3   15   3

Diff in
x values   3   2   1   2   1   5   1

x :   -5   -2   0   1   3   4   9   10

x :   -5   -2   0   1   3   4   9   10

$f(x) = 3x + 2$ -13   -4   2   5   11   14   29   32

$\left(\begin{array}{c}\text{Divided}\\ \text{Diff}\end{array}\right)$: $\frac{5-2}{1-0}$ $\frac{11-5}{3-1}$ $\frac{14-11}{4-3}$ $\frac{29-14}{9-4}$

                        3   3   3   3   3 · ~

$$x_0 \qquad x_1 \qquad \text{Div Diff.}$$

$$f(x_0) \qquad f(x_1) \qquad \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Remark: If we
assume "divided"
difference formula

	$x=-1$	$x=0$	$x=1$	$x=2$
	$x^2=1$	$x^2=0$	$x^2=1$	$x^2=4$
	1	0	1	4
Diff		-1	1	3
Diff^2	$\sim$ 2	2	2	2 $\sim\sim$
2nd Div Diff $\quad x_{n+2}-x_n$ $= 2$	$\sim\sim -$	1	1	1	1 $\sim\sim$

Go back $\quad x$	-3	-2	0	1	3	4
x^2	9	4	0	1	9	16

1st Div Diff
$$\frac{4-9}{-2-(-3)} \qquad \frac{0-4}{0-(-2)} \qquad \frac{1}{1} \qquad \frac{9-1}{3-1} \qquad \frac{16-9}{4-3}$$

$$-5 \qquad -2 \qquad 1 \qquad 4 \qquad 7$$

$$\begin{array}{ccc} x_1=1 & x_2=3 & x_1=4 \\ x_0=0 & x_1=1 & x_0=3 \end{array}$$

$$\frac{4-1}{\text{need } 3} \leftarrow x_2-x_0=3$$