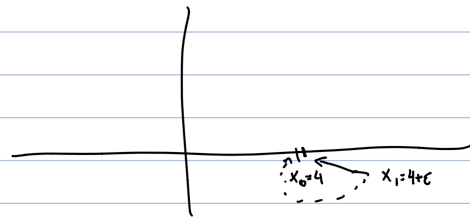
# CPSC 303:

Last time:




Data $(x_0,y_0),\ (x_1,y_1)$ with $x_0 = 4$, $x_1 = 4+\varepsilon$

$$p(x) = c_0 + x c_1$$

$$\begin{cases} c_0 + 4c_1 = y_0 \\ c_0 + (4+\varepsilon)c_1 = y_1 \end{cases} \quad \text{cond \# order } \frac{1}{\varepsilon}$$

$\varepsilon \to 0 \quad \text{cond} \to \infty$

$$?\quad \begin{cases} c_0 + 4c_1 = y_0 \\ \varepsilon c_1 = y_1 - y_0 \end{cases}$$

$$?\quad \begin{cases} c_0 + 4c_1 = y_0 \\ c_1 = (y_1 - y_0)/\varepsilon \end{cases}$$

Rem:

10.2 Polynomial Int $\quad p(x) = c_0 + c_1 x + \dots + c_n x^n$

$(x_i, y_i)$ $\quad$ functions: span $1, x, \dots, x^n$

$y_i$ varying

10.3 Langrange basis: polys = delta functions

$(x_i, y_i)$

$$i = 0,\dots,n, \quad L_i(x) = \begin{cases} 1 & \text{for } x = x_i \\ 0 & \text{for } x = x_j,\ j \neq i \end{cases}$$

say $x_0$ fixed
$x_1 = x_0 + \varepsilon,\ \varepsilon \to 0$

"bad conditioning"

10.4 $\quad p(x) = \hat{c}_0 + \hat{c}_1(x - x_0) + \hat{c}_2(x-x_0)(x-x_1) + \dots + \hat{c}_n(x-x_0)\cdots(x-x_{n-1})$

$(x_i, y_i)$ but $y_i = f(x_i)$, nice, smooth $f$

"good conditioning"

$y_i = f(x_i)$

$$\begin{cases} c_0 + 4c_1 = y_0 \\ c_0 + (4+\varepsilon)c_1 = y_1 \end{cases}$$ cond # order $\frac{1}{\varepsilon}$

$\varepsilon \to 0$ cond $\to \infty$

? $\begin{cases} c_0 + 4c_1 = y_0 \\ \varepsilon c_1 = y_1 - y_0 \end{cases}$ cond #

cond $\begin{bmatrix} 1 & 4 \\ 1 & 4+\varepsilon \end{bmatrix} \sim \frac{1}{\varepsilon}$ (constant)

$\to \infty$ bad

? $\begin{cases} c_0 + 4c_1 = y_0 \\ c_1 = (y_1 - y_0)/\varepsilon \end{cases}$ cond #

$0 < \varepsilon < 0.1$

cond $\begin{bmatrix} 1 & 4 \\ 0 & \varepsilon \end{bmatrix}$ : max entry in abs value $\begin{bmatrix} 1 & 4 \\ 0 & \varepsilon \end{bmatrix} = 4$

$$\begin{bmatrix} 1 & 4 \\ 0 & \varepsilon \end{bmatrix}^{-1} = \frac{1}{1\cdot\varepsilon - 0\cdot 4} \begin{bmatrix} \varepsilon & -4 \\ -0 & 1 \end{bmatrix},$$ max entry in abs value $\frac{1}{\varepsilon}\cdot 4$

$$4\left(\frac{4}{\varepsilon}\right) \le \text{cond}\begin{bmatrix} 1 & 4 \\ 0 & \varepsilon \end{bmatrix} \le 4\left(\frac{4}{\varepsilon}\right)\cdot n^2 = 4\left(\frac{4}{\varepsilon}\right)\cdot 4$$

$\to \infty$ as $\varepsilon \to 0$

cond $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ : max entry 4, $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}^{-1} = 1 \cdot \begin{bmatrix} 1 & -4 \\ -0 & 1 \end{bmatrix}$

is finite as $\varepsilon \to 0$.

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$\begin{cases} c_0 + 4c_1 = y_0 \\ \varepsilon c_1 = y_1 - y_0 \end{cases}$ cond # $\leftarrow$ 10.2 standard $1, x, \ldots$

10.3 Lagrange

$\begin{cases} c_0 + 4c_1 = y_0 \\ c_1 = \underbrace{(y_1 - y_0)/\varepsilon} \end{cases}$ cond # $\leftarrow$ 10.4 Newton basis

"Divided differences"

$\frac{y_1 - y_0}{\varepsilon}$ , $\varepsilon = 10^{-200}$ $(y_1 - y_0)\, 10^{200}$

but: $y_0 = f(x_0)$, $y_1 = f(x_1)$
$= f(4)$, $= f(4+\varepsilon)$

$\varepsilon \to 0$

then $\frac{y_1 - y_0}{\varepsilon} = \frac{f(4+\varepsilon) - f(4)}{\varepsilon} \longrightarrow f'(4)$

but we took $p(x) = c_0 + c_1 x$ work a little

Computationally

$\varepsilon c_1 = y_1 - y_0$

or

$c_1 = \frac{y_1 - y_0}{\varepsilon}$

if these agree to 14-16 digits of precision

$\frac{0}{\varepsilon}$

$f(x) = 7x + 5$ : $c_1 = \frac{(7x_1 + 5) - (7x_0 + 5)}{\varepsilon}$

$=$

if indistinguishable to within finite precision

(when $7x_1 + 5 = 7x_0 + 5$ in finite precision)

Say: $x_0 = 4$, $x_1 = 4 + \varepsilon$, $x_2 = 5$

Cond bad

$$\begin{cases} c_0 + 4c_1 + 16c_2 = y_0 \\ c_0 + (4+\varepsilon)c_1 + (4+\varepsilon)^2 c_2 = y_1 \\ c_0 + 5c_1 + 25c_2 = y_2 \end{cases}$$

$\varepsilon \to 0$ :

cond bad

$$\begin{cases} c_0 + 4c_1 + 16c_2 = y_0 \\ \varepsilon c_1 + (8\varepsilon + \varepsilon^2)c_2 = y_1 - y_0 \end{cases}$$

$(4+\varepsilon)^2 = 16 + 8\varepsilon + \varepsilon^2$

cond good

$$\begin{cases} c_0 + 4c_1 + 16c_2 = y_0 \\ c_1 + (8+\varepsilon)c_2 = \dfrac{y_1 - y_0}{\varepsilon} \\ c_0 + 5c_1 + 25c_2 = y_2 \end{cases}$$

$$\text{cond}\begin{bmatrix} 1 & 4 & 16 \\ & 1 & 8+\varepsilon \\ 1 & 5 & 25 \end{bmatrix} \xrightarrow{\varepsilon \to 0} \text{cond}\begin{bmatrix} 1 & 4 & 16 \\ & 1 & 8 \\ 1 & 5 & 25 \end{bmatrix}$$