CPSC 303: ($p \geq 1$, especially $p = 1, 2, \infty$)

$\mathrm{cond}_p(A) \overset{\text{(definition)}}{:=} \max_{\hat{b} \neq \vec{b}} \dfrac{\mathrm{Rel}_p(\hat{x}, x)}{\mathrm{Rel}_p(\hat{b}, b)}$ $\left\{ \begin{array}{l} A\vec{x} = \vec{b} \\ A\hat{x} = \hat{b} \end{array} \right.$

$= \max_{\hat{b} \neq \vec{b}} \dfrac{\mathrm{Rel}_p(A^{-1}\hat{b}, A^{-1}\vec{b})}{\mathrm{Rel}(\hat{b}, \vec{b})}$

Rel_p

(also "Black Box")

Magic!

= maximum loss of relative p-norm error in "solving $A\vec{x} = \vec{b}$"

Theorem: $\mathrm{cond}_p(A) = \underbrace{\|A\|_p \, \|A^{-1}\|_p}_{\text{fairly easy to approximate}}$ (p-norm of matrices)

Section 5.8 of [A & G]: $n = 10, 100,$ or $10^4, 10^5, \ldots$

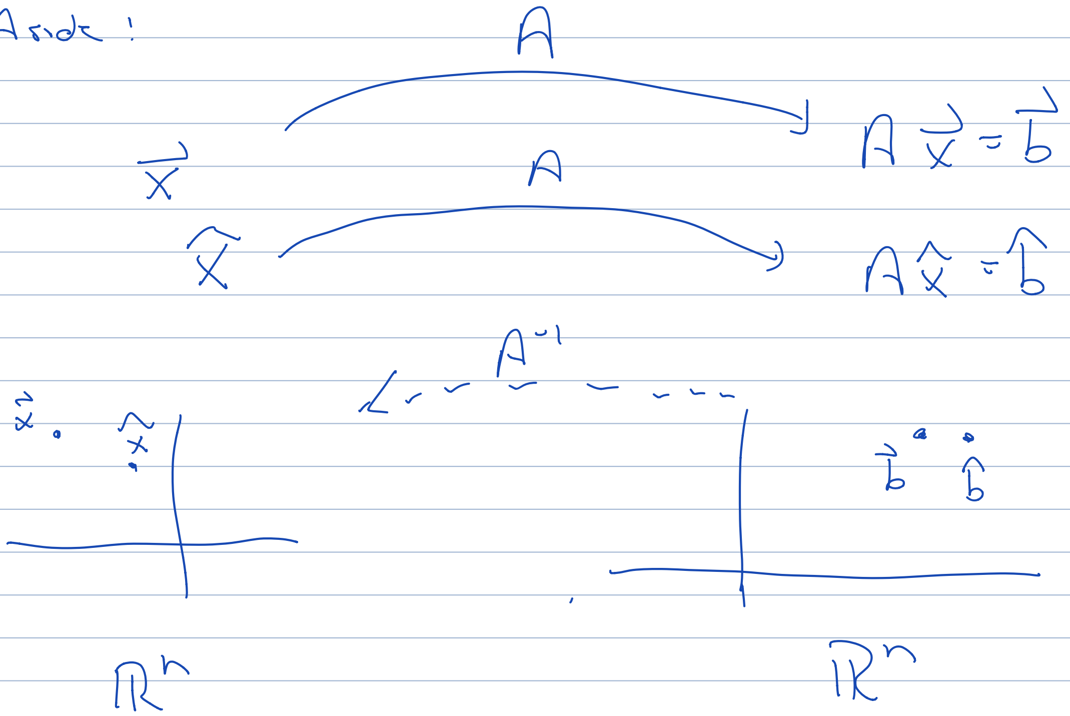
Us: $n = 2, 3,$ (Ch 11) 4

Fact: $\mathrm{cond}_p(A) = \|A\|_p \, \|A^{-1}\|_p \geq 1$

We fix $p \geq 1$ ($p = \infty, 1, 2$), A is $n \times n$

$$\|A\|_p = \max_{\vec{v} \neq 0} \frac{\|A\vec{v}\|_p}{\|\vec{v}\|_p}$$ } how much A "streches a vector" in p-norm

Aside:



Good news:

Say $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $M = \max(|\alpha|, |\beta|, |\gamma|, |\delta|)$

then

$$\boxed{M \le \|A\|_p \le 2M \quad (\text{any } p \ge 1)}$$

"black-box" magic thm

Last time:

$$\begin{bmatrix} ? & ? \\ 27 & ? \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{p\text{-norm} = 1} = \underbrace{\begin{bmatrix} ? \\ 27 \end{bmatrix}}_{p\text{-norm} \ge 27}$$

Exact 1-norm, 2-norm, ∞-norm?

$$\left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\|_\infty = \max(|\alpha| + |\beta|, |\gamma| + |\delta|)$$

$$\left\| \qquad \right\|_1 = \max(|\alpha| + |\gamma|, |\beta| + |\delta|)$$

$$\| \quad \|_2 = \sqrt{\text{largest eigenvalue of } A^T A}$$

I initially goofed until corrected by student

Aside!

$$\left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\|_\infty = ?$$

$$\max_{\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \vec{0}} \frac{\left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty}{\left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty}$$

True for any norm

$$= \max_{\left\| \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty = 1} \left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty$$

$$\max_{\left\| \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty \leq 1} \left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty$$

$$= \max_{|v_1|, |v_2| \leq 1} \left(\max \left(\alpha v_1 + \beta v_2 , \ \gamma v_1 + \delta v_2 \right) \right)$$

$|v_1| \leq 1$ $\quad$ $|v_2| \leq 1$

Say

$$A = \begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix} \; :$$

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix} \leftarrow \max_{\infty} \; 9$

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \leftarrow \text{''} < 9$

$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \qquad < 9$

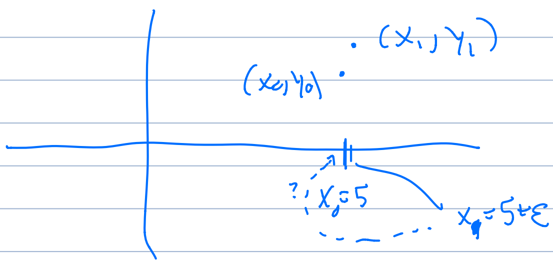
$A\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \qquad < 9$

So

$$\|A\|_\infty = \max\left(|5| + |-4|,\; |1| + |2|\right)$$

Interpolation: Tangent Line

e.g. $x_0 = 5$, $x_1 = 5 + \varepsilon$, ε small



$$p(x) = c_0 + c_1 x$$

$$c_0 + 5c_1 = y_0$$

$$c_0 + (5+\varepsilon)c_1 = y_1$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 5 \\ 1 & 5+\varepsilon \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 \\ 1 & 5+\varepsilon \end{bmatrix}, \qquad A^{-1} = \frac{1}{1\cdot(5+\varepsilon) - 1\cdot 5} \begin{bmatrix} 5+\varepsilon & -5 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\varepsilon} \begin{bmatrix} \end{bmatrix}$$

$$\mathrm{cond}_p(A) = \underbrace{\left\| \begin{bmatrix} 1 & 5 \\ 1 & 5+\varepsilon \end{bmatrix} \right\|_p}_{\text{max entry}} \quad \underbrace{\left\| \frac{1}{\varepsilon} \begin{bmatrix} 5+\varepsilon & -5 \\ -1 & 1 \end{bmatrix} \right\|_p}$$

max entry

$\underbrace{5 \text{ or } |5+\varepsilon|}$

$\frac{1}{\varepsilon}\max(5, 5+\varepsilon) \leq$ norm

$\leq \frac{2}{\varepsilon}\max(5, 5+\varepsilon)$

$\max(5, 5+\varepsilon) \leq$ norm $\leq 2\max(5, 5+\varepsilon)$

As $\varepsilon \to 0$ $\quad \mathrm{cond}\begin{pmatrix} 1 & 5 \\ 1 & 5+\varepsilon \end{pmatrix}$ roughly $\frac{\text{const}}{\varepsilon}$ $\cdot 25$ $\quad 4\cdot(5.01)^2$

within factor of 4

$0 < \varepsilon < 0.1$:

$5 \leq$ first norm $\leq 2 \cdot 5.01$ $\qquad \frac{5}{\varepsilon} \leq$ second norm $\leq 2 \cdot \frac{5.01}{\varepsilon}$

$$\frac{25}{\varepsilon} \leq \mathrm{cond}(A) \leq 4\cdot(5.01)^2 / \varepsilon$$

$$\begin{cases} 1\, c_0 + 5\, c_1 = y_0 \\ 1\, c_0 + (5+\varepsilon)\, c_1 = y_1 \end{cases}$$

$$\begin{cases} 1 \cdot c_0 + 5 c_1 = y_0 \\ \varepsilon c_1 = y_1 - y_0 \end{cases}$$

cond # good or bad as $\varepsilon \to 0$

$$\boxed{\begin{cases} 1 \cdot c_0 + 5 c_1 = y_0 \\ c_1 = (y_1 - y_0)/\varepsilon \end{cases}}$$