CPSC 303, Jan 27. [A&G]

- ℓ^p-norms (or p-norms), $p = 1, 2, \infty$ §4.2

- ℓ^p-norms of a matrix

- ℓ^p-condition number §5.8

$$\boxed{\mathrm{cond}_p(A) := \max_{b \neq \hat{b}} \frac{\mathrm{Rel}_p(\hat{x}, x)}{\mathrm{Rel}_p(\hat{b}, b)} \quad \left.\begin{matrix} Ax = b \\ A\hat{x} = \hat{b} \end{matrix}\right\}}$$

(Thm) $\mathrm{cond}_p(A) = \|A\|_p \, \|A^{-1}\|_p$

Real goal: understand degenerating interpolation:

(x_0, y_0), (x_1, y_1), ... $x_1 = x_0 + \varepsilon$ very small

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Rel Error: $\underbrace{(4 \pm 0.04)}_{\text{rel err } 1\%} + (3 \pm 0.03) = 7 \pm 0.07$

(something within 1% of $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$)

length of an elt of $\mathbb{R}^2$

$$\left(\mathrm{length}_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \sqrt{|v_1|^2 + |v_2|^2} \leftarrow p = 2$$

$$\text{length}_p \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(|v_1|^p + |v_2|^p \right)^{1/p}$$

$p = 1$ $$\text{length}_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_1 = |v_1| + |v_2|$$

"$p = \infty$" $$\left.\begin{aligned} \text{length}_\infty \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_{max} \end{aligned}\right\} \max(|v_1|, |v_2|)$$

Require a "norm" ("length", "magnitude")

function $\mathbb{R}^2$

$\mathbb{R}^n \longrightarrow$ non-negative, real number

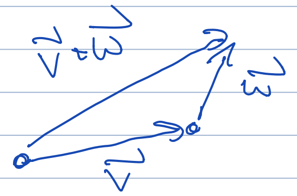
$\mathbb{C}^n$

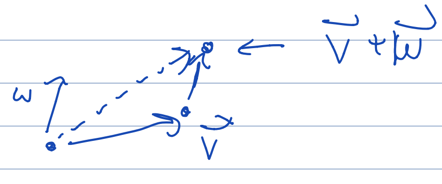
vector sp

(1) $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ iff $\vec{v} = \vec{0}$

(2) $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$, $\alpha \in \mathbb{R}$, $\vec{v}$

(3) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$





$\| \ \|_1$, $\| \ \|_2$, $\| \ \|_\infty = \| \ \|_{max}$, $\| \ \|_p$, ...

Fix some norm, say $\|\ \|_p$, $p = 1, 2, \infty$

Take

something within 1% of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

something " " " $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

??? ? ?

Claim: within 1% of $\begin{bmatrix} 12 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\quad \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \le 0.01 \left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\|$

$\begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\quad \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \le 0.01 \left\| \begin{bmatrix} 8 \\ 2 \end{bmatrix} \right\|$

$\begin{bmatrix} 12 \\ 3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\quad \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \overset{\text{triangle}}{\le} \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| + \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|$

$$\le (0.01) \left(\left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 8 \\ 2 \end{bmatrix} \right\| \right)$$

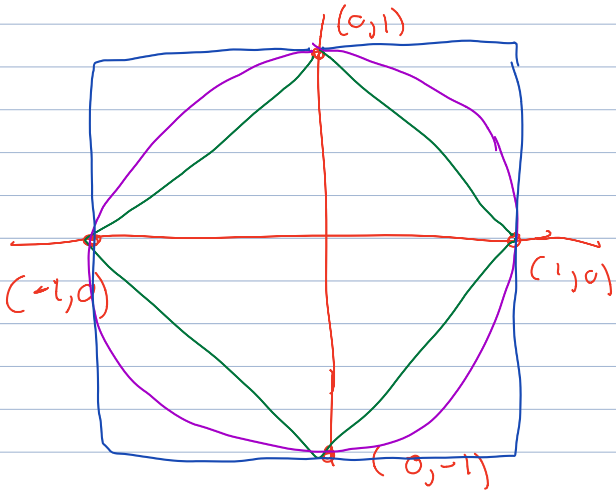
$$3 \left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\| \longleftarrow \left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\| + 2 \left\| \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 12 \\ 3 \end{bmatrix} \right\|$$

But! $\left(\text{within } 1\% \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) + \left(\text{within } 1\% \text{ of } \begin{bmatrix} -4 \\ -1 \end{bmatrix} \right) =$ ☹

Moral: Even $n = 1$ tells you about problems with relative error ...

Last time:



vectors of length 1

$(0, \pm 1)$
$(\pm 1, 0)$ length 1 in p-norm

$\| \ \|_1$

$\| \ \|_2$

$\| \ \|_\infty$

Fact $\quad \|\vec{v}\|_\infty \leq \|\vec{v}\|_2 \leq \sqrt{n}\, \|\vec{v}\|_\infty$

$\|\vec{v}\|_\infty \leq \|\vec{v}\|_1 \leq n\, \|\vec{v}\|_\infty$

For $n \times n$ matrix:

$$\|A\|_p = \max_{\vec{v} \neq 0} \frac{\|A\vec{v}\|_p}{\|\vec{v}\|_p}$$

e.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix},$$ claim: for any $p = 1, 2, \infty$ $(p \geq 1)$

$$\left\| \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right\|_p = 3 = \text{most } \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

can "stretch" a vector in terms of p-norm

$$\left\| \begin{bmatrix} 10 & 11 \\ 12 & 13 \end{bmatrix} \right\|_2 = \max \frac{\left\| \begin{bmatrix} 10 & 11 \\ 12 & 13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_2}$$

$$= ???$$

Exact answer (2-norm) largest eigenvalue of AA^T

square root of the

" " " $\begin{bmatrix} 10 & 11 \\ 12 & 13 \end{bmatrix} \begin{bmatrix} 10 & 12 \\ 11 & 13 \end{bmatrix}$

Claim: If M is the largest entry of A, A is $n \times n$

$$M \leq \|A\|_p \leq n \cdot M$$

$$M \leq \qquad \leq 2M \quad \leftarrow n=2$$

$$=$$

$$\mathrm{cond}_p(A) = \begin{cases} \|A\|_p \, \|A^{-1}\|_p \\ = \\ \max \dfrac{\mathrm{Rel}_p(\hat{x}, x)}{\mathrm{Rel}_p(\hat{b}, b)} \end{cases} \quad \begin{cases} Ax = b \\ A\hat{x} = \hat{b} \end{cases}$$

$$13 \leq \left\| \begin{bmatrix} 10 & 11 \\ 12 & 13 \end{bmatrix} \right\|_2 \leq 26$$

$$\begin{bmatrix} 10 & 11 \\ 12 & 13 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix} = \begin{bmatrix} ? \\ 13 \end{bmatrix}$$

length 1   length ≥ 13

$$\begin{bmatrix} 0 & 44 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 44 \\ ? \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 96 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{?} \\ 96 \end{bmatrix} \longleftarrow \text{length} \geq 96$$

even in the p-norm length 1

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_2 \geq m,$$

$$m = \max(|a|, |b|, |c|, |d|)$$

$$\left\| \begin{bmatrix} 13 \\ 7 \\ 91 \end{bmatrix} \right\|_{2020} = \left(13^{2020} + 7^{2020} + 91^{2020}\right)^{1/2020}$$

$$\leq \left(3 \cdot 91^{2020}\right)^{1/2020}$$

$$\leq 3^{1/2020} \cdot 91$$

$$= 3^{1/2020} \cdot \text{max comp of}$$

$$= 3^{1/2020} \left\| \begin{bmatrix} 13 \\ 7 \\ 91 \end{bmatrix} \right\|_\infty$$

$\vec{v} \in \mathbb{R}^n$

$$\|\vec{v}\|_p \leq n^{1/p} \|\vec{v}\|_\infty$$

So

$n = 2$ diff $\| \ \|_1, \| \ \|_2, \| \ \|_\infty$

no more than a factor of 2

$$\left\| \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \right\|_p = |a_1| \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\|_p = |a_1|$$