CPSC 303: Today: ℓ^p-norms and condition numbers

[A&G], Sections 4.2 & 5.8

CPSC 302: Numerical Lin Alg
" 303: Interp., Diff, Int., ODE's, PDE's, ...

} Lin Alg is behind everything

common to everything:
relative error, condition numbers

ℓ^p-norms

Most everything we need to know will be illustrated in dims $n=1,2,3$.
(for CPSC 303: interpolation, etc.)

Relative error in solving $A\vec{x}=\vec{b}$, A is $n\times n$ system
A invertible

$$\boxed{\text{cond}(A) = \text{the worst case loss of "relative error" from } \vec{b} \text{ to } \vec{x}}$$

Idea: $n=1$:

$$731\,x = b = 7 \pm 1\%$$
$$= 7 \pm 0.07$$

Techically: $b \in [6.93, 7.07]$, i.e. $|b-7| \le 0.07$

$$\Rightarrow \quad x = \frac{1}{731}(7 \pm 0.07)$$ ← determines x to within 1%

More generally, if b is known to within any $\begin{Bmatrix} \text{percent} \\ \text{relative error} \end{Bmatrix}$

then so is x for $Ax=b$, and $n=1$.

Say $n=2$:
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix} \right.$$

Aside: $(4 \pm 0.04) + (3 \pm 0.03) = 7 \pm 0.07$

$(4 \pm 0.04) + (-3 \pm 0.03) = 1 \pm 0.07$

0.07 = absolute error, the same

relative error: $\frac{0.07}{7} = 1\%$, $\frac{0.07}{1} = 7\%$

$(4 \pm 0.04) + (-4 \pm 0.04) = 0 \pm 0.08 \rightarrow$ " $\frac{0.08}{0} = \infty$ "

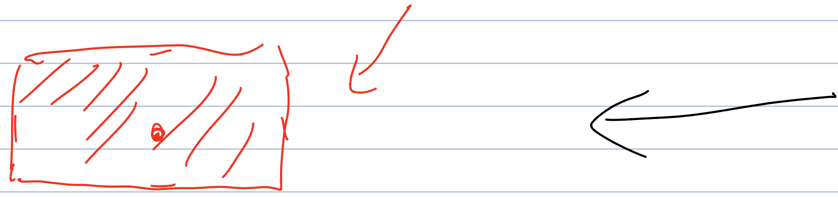
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix} = \frac{1}{1\cdot 4 - 2\cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix} = 4(6 \pm 0.06)$$
$$+(-2)(7 \pm 0.07)$$

$$= 10 \pm 0.38$$

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \pm 0.06 \\ 7 \pm 0.07 \end{bmatrix} = \begin{bmatrix} 10 \pm 0.38 \\ -11 \pm 0.25 \end{bmatrix}$$

-18
7



$(10, -11)$

In $\mathbb{R}^2$:

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \ \mathbb{C}^n \qquad \mathbb{R} = \text{reals}, \ \mathbb{C} = \text{complex}$$

most popular $\left\{ \begin{array}{l} \text{norms} \\ \text{lengths} \\ \text{magnitudes} \end{array} \right\}$

sometimes easier → to work with

$$\|\vec{v}\|_2 = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2} \qquad \text{2-norm}$$

$$\|\vec{v}\|_1 = |v_1| + |v_2| + \dots + |v_n| \qquad \text{1-norm}$$

easier for cond num

$$\|\vec{v}\|_\infty = \max(|v_1|, \dots, |v_n|) \qquad \text{max-norm, } \infty\text{-norm}$$

$$\|\vec{v}\|_p = \left(|v_1|^p + |v_2|^p + \dots + |v_n|^p \right)^{1/p} \qquad p \geq 1 \quad (0 < p < 1)$$

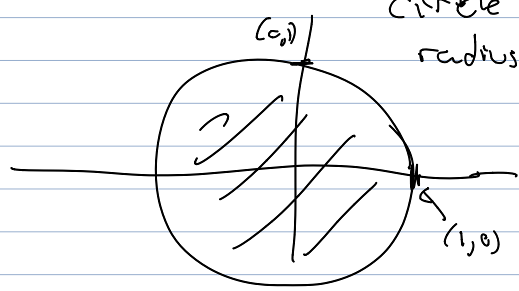
$$\|\vec{v}\|_p \xrightarrow{p \to \infty} \qquad (p = 1, 2, \infty)$$

$$\left\|\begin{bmatrix}1\\2\\7\end{bmatrix}\right\|_p = \Big(\underbrace{1^p + 2^p + 7^p}_{\left(\substack{\text{between}\\ 1 \text{ and } 3}\right) 7^p}\Big)^{1/p} = (7^p)^{1/p}\left(\substack{\text{between}\\ 1 \text{ and } 3}\right)^{1/p} \xrightarrow{p\to\infty} 7$$

$p=1$ $\quad 1+2+7$

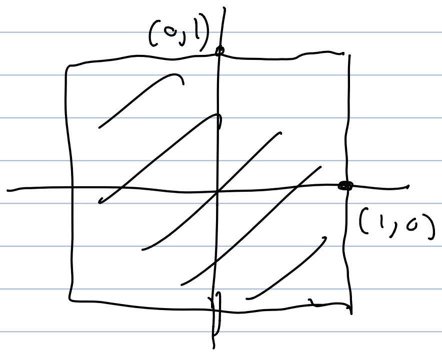
$p=2$ $\quad \sqrt{1^2+2^2+7^2}$

$p=\infty$ $\quad 7$



circle of radius 1

$\| \ \|_2$



vectors in $\mathbb{R}^2$ of $\| \ \|_\infty \le 1$

$\{ \|x\|_1 \le 1 \}$