CPSC 303: Plan next 2-4 classes

- "Linear Algebra w/o Linear Algebra"
  in interpolation and beyond...

- Degenerate interpolation, e.g. x_0, x_1 "very close"
  (condition numbers & derivatives)

- 3 bases: monomial, Lagrange, Newton

Lecture Notes 2: Polynomial Interpolation

CPSC 303: Numerical Approximation and Discretization

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Outline

- Background
  - Problem statement and motivation
  - Formulation: The linear system and its conditioning
- Polynomial bases
  - Monomial
  - Lagrange
  - Newton
  - Uniqueness of polynomial interpolant
- Divided differences
  - Divided difference tables and the Newton basis interpolant
  - Divided difference connection to derivatives
- Osculating interpolation: interpolating derivatives
- Error analysis for polynomial interpolation
  - Reducing the error using the Chebyshev points as abscissae

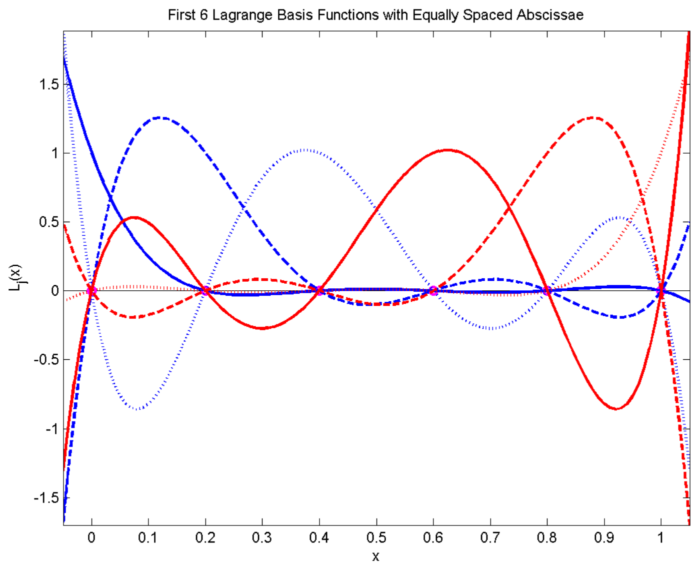
Interpolation Motivation

We are given a collection of **data samples** $\{(x_i, y_i)\}_{i=0}^n$

- The $\{x_i\}_{i=0}^n$ are called the **abscissae** (singular: abscissa), the $\{y_i\}_{i=0}^n$ are called the **data values**
- Want to find a function $p(x)$ which can be used to estimate $y(x)$ for $x \neq x_i$
- Why? We often get discrete data from sensors or computation, but we want information as if the function were not discretely sampled
- If possible, $p(x)$ should be inexpensive to evaluate for a given x

Lagrange Basis Conditioning

The Lagrange basis functions $L_j(x)$ are clearly distinct. With abscissae $x_i = {}^i\!/_5$ for $i = 0, 1, \ldots, 5$ the Lagrange basis function $L_j(x)$ are shown below.



Basic Polynomial Interpolation Summary

- We want to interpolate data $\{(x_i, y_i)\}_{i=0}^n$ using basis function set $\{\phi_j(x)\}_{j=0}^n$
  - Interpolant is $p_n(x) = \sum_{j=0}^n c_j \phi_j(x)$
  - Interpolation conditions $p_n(x_i) = y_i$ lead to square linear system $Ac = y$, where $a_{ij} = \phi_j(x_i)$
  - Solve linear system for coefficients c_j
- Interpolating polynomial is unique, but choice of basis set affects cost of construction and evaluation, accuracy, and difficulty of changing or adding data

basis set	construction cost	evaluation cost	bonus feature
monomial	$(1/3)n^3 + \mathcal{O}(n^2)$	n	simple
Lagrange	$n^2 + \mathcal{O}(n)$	$3n$	$c_j = y_j$
Newton	$n^2 + \mathcal{O}(n)$	n	adaptive n

Linear Algebra w/o Linear Algebra:

Say we want to find

$$v(x) = c_0 + c_1 x + c_2 x^2$$

to fit data: $(2, y_0), (3, y_1), (5, y_2)$

Any solution, essentially solving

$$\begin{aligned} c_0 + 2c_1 + 4c_2 &= y_0 \\ c_0 + 3c_1 + 9c_2 &= y_1 \\ c_0 + 5c_1 + 25c_2 &= y_2 \end{aligned}$$

Does this have a unique solution?

<u>Theory</u>: 3×3 system has unique sol $\Leftrightarrow$ homog form has unique solution:

$$\left.\begin{aligned} c_0 + 2c_1 + 4c_2 &= 0 \\ c_0 + 3c_1 + 9c_2 &= 0 \\ c_0 + 5c_1 + 25c_2 &= 0 \end{aligned}\right\} \text{unique solution?}$$

Is there a unique c_1, c_2, c_3 (namely $c_1, c_2, c_3 = 0$) s.t.

$$v(x) = c_0 + c_1 x + c_2 x^2$$

has $v(2), v(3), v(5) = 0$?

is there unique $V(x)$ s.t. $v(x)$ models
$(2, 0)$, $(3, 0)$, $(5, 0)$

If $v(x)$, deg ≤ 2 polynomial has roots
$x=2, x=3, x=5$, then $\Rightarrow$ $v(x)$ must be 0

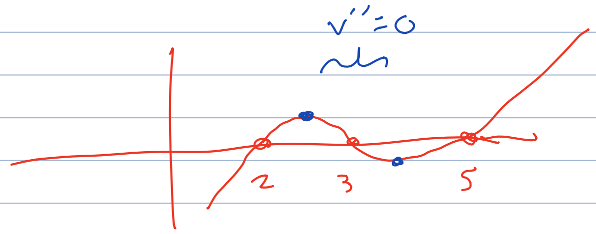
(1) $v(x) = (x-2)(x-3)(x-5)$ some other poly

$\Rightarrow$ must have $v(x) = 0$

(2) $v(x)$ has roots at $2, 3, 5$

Rolle's thm: $v'(x)$ has roots $(2, 3)$
$(3, 5)$

$v''(x)$ " " between $2, 5$



$v(x) = c_0 + c_1 x + c_2 x^2$

$v''(x) = 2 \cdot c_2$

$\Rightarrow c_2 = 0$

$v(x) = c_0 + c_1 x$, so v' has a root, $c_1 = 0$

$\Downarrow$

$c_1 \longrightarrow c_1 = 0$

similarly $c_0 = 0$

Say $v(x) = \varphi_0(x)\, c_0 + \varphi_1(x)\, c_1 + \varphi_2(x)\, c_2$

fit $(x_0, y_0), (x_1, y_1), (x_2, y_2)$:

$$\begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_0(x_2) & \varphi_1(x_2) & \varphi_2(x_2) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

has unique solution

$\Leftrightarrow$ $v(x_0) = 0,\ v(x_1) = 0,\ v(x_2) = 0$ get a unique sol, $v(x) = 0$

If $x_0 = x_1$ then 2 rows same;

what happens if $(x_0, y_0),\ \underbrace{(x_1, y_1)}_{=(x_0,\, y_1)},\ (x_2, y_2)$

Say polynomials: $\left(\begin{matrix} x_0 = x_1 = 2, \\ x_2 = 5 \end{matrix}\right)$

$$\left.\begin{aligned} c_0 + 2c_1 + 4c_2 &= 0 \\ c_0 + 2c_1 + 4c_2 &= 0 \\ c_0 + 5c_1 + 25c_2 &= 0 \end{aligned}\right\} \text{unique solution?}$$

What if x_0, x_1 are very near 2 ?

$$c_0 + 2c_1 + 4c_2 = 0$$
$$c_0 + 2.1c_1 + (2.1)^2 c_2 = 0$$
$$c_0 + 5c_1 + 25c_2 = 0$$

$2.1 \to 2.01 \to 2.001 \to$ etc

Before this, $n=1$, $V(x) = c_0 + x c_1$

fit $(2, y_0)$, $(2+\varepsilon, y_1)$ $\qquad \varepsilon \to 0.1,\ 0.001,\ 0.00001,\ \ldots$

Solve

"condition number"

$$\left.\begin{aligned} c_0 + 2c_1 &= y_0 \\ c_0 + (2+\varepsilon)c_1 &= y_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} c_0 + 2c_1 &= y_0 \\ + \varepsilon c_1 &= y_1 - y_0 \end{aligned}\right\}$$

$$\left.\begin{aligned} c_0 + 2c_1 &= y_0 \\ c_1 &= (y_1 - y_0)/\varepsilon \end{aligned}\right\}$$

Say differentiable $f(x)$, fit data

$$(x_0, \underbrace{f(x_0)}_{y_0}), \quad (x_0+\varepsilon, \underbrace{f(x_0+\varepsilon)}_{y_1})$$

$f'(x_0)$

Next time --- "condition number"