CPSC 303: MIDTERM PRACTICE QUESTIONS, SET 3

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All questions on Homework 1–6 should be considered midterm practice.

(1) Define for each sequence
$$\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}} = \{\ldots, y_{-1}, y_0, y_1, y_2, \ldots\}$$
a new sequence $D\mathbf{y}$ (known as the "(forward) difference of $\mathbf{y}$") defined by
$$(D\mathbf{y})_n = y_{n+1} - y_n.$$

(a) Show that if $\mathbf{y}$ is given by $y_n = n^3$, then $D\mathbf{y}$ is a polynomial in n of degree 2 and leadning term $3n^2$. **Solution:**
$$(D\mathbf{y})_n = y_{n+1} - y_n = (n+1)^3 - n^3$$
$$= \Bigl(n^3 + 3n^2 + 3n + 1\Bigr) - n^3 = 3n^2 + \textbf{ lower order terms}$$

(b) Show that if $\mathbf{y}$ is given by $y_n = n^4$, then $D\mathbf{y}$ is a polynomial in n of degree 3 and leadning term $4n^3$. **Solution:**
$$(D\mathbf{y})_n = y_{n+1} - y_n = (n+1)^4 - n^4$$
$$= \Bigl(n^4 + 4n^3 + 6n^2 + \textbf{ lower order terms }\Bigr) - n^4 = 4n^3 + \textbf{ lower order terms}$$

(c) Show that if $\mathbf{y}$ is given by $y_n = n^4$, then $D^2\mathbf{y}$ is a polynomial in n of degree 2 and leadning term $12n^2$. **Solution:**
$$D^2\mathbf{y} = D(D\mathbf{y}) = D(4n^3 + \textbf{ lower order terms })$$
$$= 4Dn^3 + D(\textbf{ lower order terms }) = 4(3n^2) + (\textbf{ lower order terms })$$

(2) Define for each sequence
$$\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}} = \{\ldots, y_{-1}, y_0, y_1, y_2, \ldots\}$$
a new sequence $D\mathbf{y}$ (known as the "(forward) difference of $\mathbf{y}$") defined by
$$(D\mathbf{y})_n = y_{n+1} - y_n.$$

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(a) Show that $D^3\mathbf{y}$ is the sequence whose n-th term is

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n.$$

Solution: See Homework 4.

(b) Show that the sequences $\mathbf{y}$ given by either $y_n = 1$ or $y_n = n$ or $y_n = n^2$ all satisfy $D^3\mathbf{y} = 0$. **Solution: See Homework 4.**

(c) Show that for any $C_1, C_2, C_3 \in \mathbb{R}$, the sequence $\mathbf{y}$ given by either $y_n = C_1 + C_2 n + C_3 n^2$ all satisfy $D^3\mathbf{y} = 0$. **Solution: See Homework 4.**

(d) Show that the sequence $\mathbf{y}$ given by $y_n = n^3$ satisfies $D^4\mathbf{y} = 0$. **Solution: We have $D\mathbf{y}$ is a polynomial of degree 2 in n (see the Problem 1 above). Hence D^3 applied to $D\mathbf{y}$ is zero (see part (a) above).**

(e) Show that for any C_1, C_2, C_3, C_4 the sequence given as $y_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$ satisfies $D^4\mathbf{y} = 0$. **Solution: If $y_n = C_1 + C_2 n + C_3 n^2$, then $D^3\mathbf{y} = 0$, so $D^4\mathbf{y} = D(D^3\mathbf{y}) = 0$. In addition $y_n = n^3$ also satisfies $D^4\mathbf{y} = 0$; hence $C_4 n^3$ is another solution to $D^4\mathbf{y} = 0$, and hence so is**

$$y_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$$

(f) Explain why for any y_0, y_1, y_2, y_3 there exist $C_1, \ldots, C_4$ that satisfy

$$y_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3$$

for $n = 0, 1, 2, 3$. **Solution: We know that there is a polynomial, p, of degree at most 3 that interpolates the 4 data points**

$$(0, y_0),\ (1, y_1),\ (2, y_2),\ (3, y_3).$$

Writing $p(x)$ as $C_1 + C_2 x + C_3 x^2 + C_4 x^3$, we get the desired constants C_1, C_2, C_3, C_4.

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