CPSC 303: MIDTERM PRACTICE QUESTIONS, SET 2, SOLUTIONS TO SOME QUESTIONS

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All questions on Homework 1–6 should be considered midterm practice.

(1) True/False

(a) The p-norm of the matrix
$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$
is $|a| + |b|$ for $p = 1$, $p = 2$, and $p = \infty$. **Solution: True**

(b) The ∞-norm of a matrix is the maximum sum of the absolute values of the entries in any row. **Solution: True**

(c) The 1-norm of a matrix is the maximum sum of the absolute values of the entries in any column. **Solution: True**

(d) The 2-norm of any nonzero matrix is positive. **Solution: True: indeed, the p-norm is bounded below by M (and above by nM) where M is the largest absolute value of an entry of the matrix (for any p); if a matrix is nonzero then $M > 0$, so its p-norm is positive for any p.**

(e) The ∞-norm of any nonzero matrix is positive. **Solution: True (see the question above).**

(f) There is a unique polynomial $p(x)$ of degree at most 3 such that $p(1) = 2020$, $p'(1) = 2021$, $p''(1) = 2022$. **Solution: False: this gives three linear equations for the four coefficients of p.**

(g) There is a unique polynomial $p(x)$ of degree at most 3 such that $p(1) = 2020$, $p'(1) = 2021$, $p''(1) = 2022$, $p(2) = 303$. **Solution: True, see 10.7 of [A&G]**

(2) **Circle the correct numeral (i, ii, iii, or iv) in each of the following questions.**

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(a) For any real a, b, the p-norm of

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

equals $|a| + |b|$ for
  (i) $p = 1$ only
  (ii) $p = 2$ only
  (iii) $p = \infty$ only
  (iv) all of the above

Solution: iv

(b) If A is an $n \times n$ matrix, and M is the largest absolute value of an entry of A, then the inequality

$$M \leq \|A\|_p \leq nM$$

holds for
  (i) $p = 1$ only
  (ii) $p = 2$ only
  (iii) $p = \infty$ only
  (iv) all of the above

Solution: iv

(c) If A is an $n \times n$ matrix diagonal matrix with diagonal entries $d_1, \ldots, d_n$, then

$$\|A\|_p = \max\bigl(|d_1|, \ldots, |d_n|\bigr)$$

holds for
  (i) $p = 1$ only
  (ii) $p = 2$ only
  (iii) $p = \infty$ only
  (iv) all of the above

Solution: iv

(3) For $p = 1, 2, \infty$, find the p-condition number (i.e., p-norm condition number) of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Solution: For any p, the p-norm of this diagonal matrix is the absolute value of its largest diagonal entry, i.e., 4, and the p-norm of its inverse,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

is, similarly, 1. Hence the p-condition number of this matrix, for any p, is $4 \cdot 1 = 4$.

(4) Use the "error in polynomial interpolation" to bound from above the error in interpolating $\sin x$ in $[0,2]$ by the polynomial $p_2(x)$ that interpolates $\sin x$ at the three points $0,1,2$.

(5) Use the "error in polynomial interpolation" to bound from above the error in interpolating x^3 in $[0,2]$ by the polynomial $p_2(x)$ that interpolates x^3 at the three points $0,1,2$. Is your bound from above tight? **Solution: Let $f(x)=x^3$. The error in interpolation formula implies that for any $x\in[0,2]$ there is a $\xi\in[0,2]$ such that the error is**

$$\frac{|f'''(\xi)|}{6}\max_{x\in[0,2]}\Bigl|(x-0)(x-1)(x-2)\Bigr|.$$

Since $f'''(\xi)=6$ for all ξ, we have that the error is exactly

$$\max_{x\in[0,2]}\Bigl|(x-0)(x-1)(x-2)\Bigr|.$$

To find the maximum absolute value of

$$v(x)=(x-0)(x-1)(x-2)=x^3-3x^2+2x$$

we compute $v'(x)=3x^2-6x+2$ and set $v'(x)=0$, and we find $x=1\pm\sqrt{3}/3$, and we calculate

$$\left|v\Bigl(1+\sqrt{3}/3\Bigr)\right|=\left|\Bigl(1+\sqrt{3}/3\Bigr)\Bigl(\sqrt{3}/3\Bigr)\Bigl(-1+\sqrt{3}/3\Bigr)\right|=\Bigl(\sqrt{3}/3\Bigr)\left|\Bigl((\sqrt{3}/3)^2-1\Bigr)\right|=2\sqrt{3}/9,$$

and similarly for $v(x)$ with $x=1-\sqrt{3}/3$. Hence

$$\max_{x\in[0,2]}\Bigl|(x-0)(x-1)(x-2)\Bigr|=2\sqrt{3}/9.$$

Furthermore, this bound happens to be exact, since f''' is a constant.

(6) Use the "error in polynomial interpolation" to bound from above the error in interpolating x^2 in $[0,2]$ by the polynomial $p_2(x)$ that interpolates x^2 at the three points $0,1,2$. Is your bound from above tight? **Solution: Since $f(x)=x^2$ has $f'''(x)=0$, the error in interpolation will be 0.**

Remark: of course, $p_2(x)$ is a unique polynomial of degree at most 2, and visibly x^2 interpolates itself (i.e., x^2) at $0,1,2$, so we know that $p_2(x)=x^2$ and hence p_2 makes no error in interpolating x^2.

(7) Use the formula

$$\cos(4\theta)=8\cos^4\theta-8\cos^2\theta+1$$

to write down a formula for the Chebyshev polynomial $T_4(x)$. **Solution:** $T_4(x)=8x^4-8x^2+1$

(8) Let $\epsilon\in\mathbb{R}$.

(a) Derive a formula for the secant line at $x = 4$ and $x = 4 + \epsilon$ for a function $f : \mathbb{R} \to \mathbb{R}$. **Solution: Let $p(x) = c_0 + c_1 x$ be the secant line; we have**

$$p(4) = f(4) \quad \Rightarrow \quad c_0 + 4c_1 = f(4)$$
$$p(4+\epsilon) \quad \Rightarrow \quad c_0 + (4+\epsilon)c_1 = f(4+\epsilon)$$

and so

$$c_1 = \frac{f(4+\epsilon) - f(4)}{\epsilon}$$

and $c_0 = f(4) - 4c_1$, and so

$$p(x) = f(4) - 4\frac{f(4+\epsilon) - f(4)}{\epsilon} + \frac{f(4+\epsilon) - f(4)}{\epsilon}x$$

(b) How does the divided difference $f[4, 4+\epsilon]$ relate to your formula in part (a)? **Solution: The slope c_1 is just**

$$c_1 = \frac{f(4+\epsilon) - f(4)}{\epsilon} = f[4, 4+\epsilon]$$

(c) Find the ∞-condition number of

$$\begin{bmatrix} 1 & 4 \\ 1 & 4+\epsilon \end{bmatrix}$$

Solution We have

$$\begin{bmatrix} 1 & 4 \\ 1 & 4+\epsilon \end{bmatrix}^{-1} = \frac{1}{\epsilon}\begin{bmatrix} 4+\epsilon & -4 \\ -1 & 1 \end{bmatrix}$$

Since the ∞-norm of a matrix is just the largest sum of absolute values of the entries in a row, we have that if $\epsilon > 0$ then

$$\left\| \begin{bmatrix} 1 & 4 \\ 1 & 4+\epsilon \end{bmatrix} \right\|_\infty = 1 + \max(4, |4+\epsilon|), \qquad \left\| \frac{1}{\epsilon}\begin{bmatrix} 4+\epsilon & -4 \\ -1 & 1 \end{bmatrix} \right\|_\infty = (4 + |4+\epsilon|)/|\epsilon|$$

(the sum of the absolute values in the bottom row of the second matrix is 2, which is always smaller than that of the top row) so the condition number of the matrix in question is

$$\frac{\Big(1 + \max(4, |4+\epsilon|)\Big)\Big(4 + |4+\epsilon|\Big)}{|\epsilon|}$$

Note: On an exam, I will likely restrict ϵ to be small and either positive or negative in order to simplify the above calculations and expressions.

(d) How does the condition number above relate to the equations needed to find the secant line? What does this tell you as $\epsilon \to 0$? **Solution: The matrix in part (c) describes the equations in part (a), which can be written as**

$$\begin{bmatrix} 1 & 4 \\ 1 & 4+\epsilon \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(4) \\ f(4+\epsilon) \end{bmatrix}$$

As $\epsilon \to 0$ the condition number tends to infinity, so the relative error in solving these equations tends to infinity, so you wouldn't want to directly solve these equations.

Remark: By substracting the top equation from the bottom equation and dividing the bottom equation by ϵ you get the equations

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(4) \\ f[4, 4+\epsilon] \end{bmatrix}$$

whose condition number is bounded, and which is a numerically reasonably way to solve the same equations; note these equations require you to know $f[4, 4+\epsilon]$, which tends to $f'(4)$ as $\epsilon \to 0$.

(9) Use the "divided difference derivative" formula to bound the value of:
   (a) $f[0, 1/2, 1]$ for $f(x) = e^x$; **Solution: The "divided difference derivative" formula tells us that**

   $$f[0, 1/2, 1] = f''(\xi)/2$$

   for some $\xi \in [0, 1]$. We have $f''(x) = e^x$, so for $\xi \in [0, 1]$ we have

   $$1 \le f''(\xi) \le e;$$

   hence

   $$1/2 \le f[0, 1/2, 1] \le e/2.$$

   (b) $f[0, 1/2, 1]$ for $f(x) = \sin x$;
   (c) $f[0, 1/2, 1]$ for $f(x) = x^2$;
   (d) $f[0, 2/3, 1]$ for $f(x) = x^2$; **Solution: $f''(\xi) = 2$ for all $\xi \in [0, 1]$, hence $f[0, 2/3, 1] = 2/2 = 1$; more generally, for $f(x) = x^2$, $f[x_0, x_1, x_2] = 1$ for any $x_0, x_1, x_2 \in \mathbb{R}$.**
   (e) $f[0, 2/3, 1]$ for $f(x) = x^3$;
   (f) $f[0, 1/2, 2/3, 1]$ for $f(x) = x^2$; **Solution: $f'''(\xi) = 0$ for all $\xi \in [0, 1]$, hence $f[0, 1/2, 2/3, 1] = 0$; more generally, for $f(x) = x^2$, $f[x_0, x_1, x_2, x_3] = 0$ for any $x_0, x_1, x_2, x-3 \in \mathbb{R}$.**
   (g) $f[0, 1/2, 2/3, 1]$ for $f(x) = x^5$; **Solution: $f'''(\xi) = 5 \cdot 4 \cdot 3 \cdot \xi^2$; for $\xi \in [0, 1]$ we have $0 \le \xi^2 \le 1$, and so**

   $$\frac{5 \cdot 4 \cdot 3 \cdot 0}{3!} \le f[0, 1/2, 2/3, 1] \le \frac{5 \cdot 4 \cdot 3 \cdot 1}{3!}$$

   and hence

   $$0 \le f[0, 1/2, 2/3, 1] \le 10.$$

(10) Show that there is a polynomial $p(y) = (y - y_0)(y - y_1)$ for some $y_0, y_1 \in \mathbb{R}$ such that

$$\max_{2 \le y \le 3} |p(y)| = 1/8.$$

What are y_0, y_1?

[Hint 1: Recall that the degree two Chebyshev polynomial is $T_2(x) = 2x^2 - 1$. Consider the linear map $g(x)$ taking $[-1, 1]$ to $[2, 3]$ given by $g(x) = x/2 + 5/2$, and the inverse map $g^{-1}(y) = 2y - 5$ (which therefore takes $[2, 3]$ to $[-1, 1]$).]

[Hint 2: Alternatively, you can think of parabolas symmetric about $x = 5/2$.]

SOLUTION BASED ON HINT 2: The easiest way is consider shifting down the parabola $(x-5/2)^2$, whose values on $2, 5/2, 3$ are, respectively, $1/4, 0, 1/4$; moving the parabola down by $1/8$ we get $p(x) = (x-5/2)^2 - 1/8$ is a parabola with leading term x^2 and

$$\max_{2\le y\le 3} |p(y)| = 1/8$$

(see the last two pages of class notes on Feb 14 for a similar idea over $[-1,1]$). The values of y_0, y_1 are the roots of p, which are

$$(x-5/2) = \pm\sqrt{1/8}, \quad \textbf{i.e.,} \quad y_0, y_1 = 5/2 \pm \sqrt{1/8}.$$

SOLUTION BASED ON HINT 1: Note that

$$T_2(2y-5)$$

takes (all) values between -1 and 1 as y varies over $[2,3]$. Since

$$T_2(2y-5) = 2(2y-5)^2 - 1 = 8y^2 + \textbf{ lower order terms},$$

we have that

$$p(y) = T_2(2y-2)/8$$

has leading coefficient y^2 and has

$$\max_{2\le y\le 3} |p(y)| = 1/8.$$

Since the roots of $T_2(x) = 2x^2 - 1$ are $\pm\sqrt{1/2}$, the roots y_0, y_1 of p are

$$g\bigl(\pm\sqrt{1/2}\bigr) = \frac{\pm\sqrt{1/2}}{2} + 5/2.$$

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