CPSC 303: MIDTERM PRACTICE QUESTIONS, SET 2

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All questions on Homework 1–6 should be considered midterm practice.

(1) True/False
  (a) The p-norm of the matrix
$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$
  is $|a| + |b|$ for $p = 1$, $p = 2$, and $p = \infty$.
  (b) The ∞-norm of a matrix is the maximum sum of the absolute values of the entries in any row.
  (c) The 1-norm of a matrix is the maximum sum of the absolute values of the entries in any column.
  (d) The 2-norm of any nonzero matrix is positive.
  (e) The ∞-norm of any nonzero matrix is positive.
  (f) There is a unique polynomial $p(x)$ of degree at most 3 such that $p(1) = 2020$, $p'(1) = 2021$, $p''(1) = 2022$.
  (g) There is a unique polynomial $p(x)$ of degree at most 3 such that $p(1) = 2020$, $p'(1) = 2021$, $p''(1) = 2022$, $p(2) = 303$.

(2) **Circle the correct numeral (i, ii, iii, or iv) in each of the following questions.**
  (a) For any real a, b, the p-norm of
$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$
  equals $|a| + |b|$ for
    (i) $p = 1$ only
    (ii) $p = 2$ only
    (iii) $p = \infty$ only
    (iv) all of the above

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(b) If A is an $n \times n$ matrix, and M is the largest absolute value of an entry of A, then the inequality

$$M \leq \|A\|_p \leq nM$$

holds for

(i) $p = 1$ only
(ii) $p = 2$ only
(iii) $p = \infty$ only
(iv) all of the above

(c) If A is an $n \times n$ matrix diagonal matrix with diagonal entries $d_1, \ldots, d_n$, then

$$\|A\|_p = \max\bigl(|d_1|, \ldots, |d_n|\bigr)$$

holds for

(i) $p = 1$ only
(ii) $p = 2$ only
(iii) $p = \infty$ only
(iv) all of the above

(3) For $p = 1, 2, \infty$, find the p-condition number (i.e., p-norm condition number) of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(4) Use the "error in polynomial interpolation" to bound from above the error in interpolating $\sin x$ in $[0, 2]$ by the polynomial $p_2(x)$ that interpolates $\sin x$ at the three points $0, 1, 2$.

(5) Use the "error in polynomial interpolation" to bound from above the error in interpolating x^3 in $[0, 2]$ by the polynomial $p_2(x)$ that interpolates x^3 at the three points $0, 1, 2$. Is your bound from above tight?

(6) Use the "error in polynomial interpolation" to bound from above the error in interpolating x^2 in $[0, 2]$ by the polynomial $p_2(x)$ that interpolates x^2 at the three points $0, 1, 2$. Is your bound from above tight?

(7) Use the formula

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$$

to write down a formula for the Chebyshev polynomial $T_4(x)$.

(8) Let $\epsilon \in \mathbb{R}$.

(a) Derive a formula for the secant line at $x = 4$ and $x = 4 + \epsilon$ for a function $f \colon \mathbb{R} \to \mathbb{R}$.
(b) How does the divided difference $f[4, 4 + \epsilon]$ relate to your formula in part (a)?
(c) Find the ∞-condition number of

$$\begin{bmatrix} 1 & 4 \\ 1 & 4 + \epsilon \end{bmatrix}$$

(d) How does the condition number above relate to the equations needed to find the secant line? What does this tell you as $\epsilon \to 0$?

(9) Use the "divided difference derivative" formula to bound the value of:
(a) $f[0, 1/2, 1]$ for $f(x) = e^x$;
(b) $f[0, 1/2, 1]$ for $f(x) = \sin x$;
(c) $f[0, 1/2, 1]$ for $f(x) = x^2$;
(d) $f[0, 2/3, 1]$ for $f(x) = x^2$;
(e) $f[0, 2/3, 1]$ for $f(x) = x^3$;
(f) $f[0, 1/2, 2/3, 1]$ for $f(x) = x^2$;
(g) $f[0, 1/2, 2/3, 1]$ for $f(x) = x^5$;

(10) Show that there is a polynomial $p(y) = (y - y_0)(y - y_1)$ for some $y_0, y_1 \in \mathbb{R}$ such that

$$\max_{2 \le y \le 3} |p(y)| = 1/8.$$

What are y_0, y_1?

[Hint 1: Recall that the degree two Chebyshev polynomial is $T_2(x) = 2x^2 - 1$. Consider the linear map $g(x)$ taking $[-1, 1]$ to $[2, 3]$ given by $g(x) = x/2 + 5/2$, and the inverse map $g^{-1}(y) = 2y - 5$ (which therefore takes $[2, 3]$ to $[-1, 1]$).]

[Hint 2: Alternatively, you can think of parabolas symmetric about $x = 5/2$.]

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