CPSC 303: MIDTERM PRACTICE QUESTIONS, SET 1, BRIEF SOLTIONS

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All questions on Homework 1–6 should be considered midterm practice.

(1) What is the largest normal positive number in double precision?
   Solution: $2^{1023}(2-2^{-52})$ **(or, equivalently,** $2^{1024}(1-2^{-53})$**).**

(2) What is the smallest normal positive number in double precision?
   Solution: 2^{-1022}

(3) What is the largest subnormal positive number in double precision?
   Solution: $(1-2^{-52})\cdot 2^{-1022}$

(4) What is the smallest subnormal positive number in double precision?
   Solution: 2^{-1074}

(5) In double precision, are the values of:

$$\texttt{Inf}-\texttt{Inf},\quad \texttt{Inf}+\texttt{Inf},\quad \texttt{Inf}+2020.$$

   Solution: Nan, Inf, Inf.

(6) **Circle the correct numeral (i, ii, iii, or iv) in each of the following questions.**
   (a) Normal numbers in double precision have roughly:
      (i) 530 bits of precision
      (ii) 53 bits of precision
      (iii) 530 digits of precision
      (iv) 53 digits of precision
   Solution: ii
   (b) In double precision, a number that is 10^7 (which is roughly 2^{23}) times larger than 2^{-1074} is stored with roughly
      (i) 7 bits of precision
      (ii) 7 digits of precision
      (iii) 2^7 bits of precision

Research supported in part by an NSERC grant.

(iv) 53 bits of precision

Solution: ii

(c) In double precision, a number that is roughly 2^9 times larger than 2^{-1074} is stored with roughly
  (i) 9 bits of precision
  (ii) 9 digits of precision
  (iii) 2^9 bits of precision
  (iv) 53 bits of precision

Solution: i

(d) In double precision, a number that is roughly 2^{100} times larger than 2^{-1074} is stored with roughly
  (i) 100 bits of precision
  (ii) 100 digits of precision
  (iii) 2^{100} bits of precision
  (iv) 53 bits of precision

Solution: iv

(e) In double precision, a number that is roughly 2^{100000} times larger than 2^{-1074} is stored with roughly
  (i) 100 bits of precision
  (ii) 100 digits of precision
  (iii) 2^{100} bits of precision
  (iv) none of the above

Solution: iv–it is stored as Inf

(f) In double precision, the expression 3^m number is evaluated exactly for a postive integer m iff
  (i) $3^m \leq 2^{53} - 1$
  (ii) $m \leq 53$
  (iii) $3^m \leq 2^{1024} - 1$.
  (iv) $3^m \leq 2^{1023}(2 - 2^{-52})$.

Solution: i

(7) Consider the recurrence

$$x_{n+2} = 2x_{n+1} + x_n.$$

(a) What is the general solution of this equation? Write this in the form $C_1 r_1^n + C_2 r_2^n$ for some $r_1, r_2 \in \mathbb{R}$. **Solution:** r_1, r_2 **are** $1 \pm \sqrt{2}$

(b) What is the solution to this recurrence in the case $x_0 = 1$ and $x_1 = 1 - \sqrt{2}$? **Solution:** $x_n = (1 - \sqrt{2})^n$

(c) If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1 - \sqrt{2}$, for n large what will $|x_n|$ be reported as:
  (i) Inf
  (ii) 0 or some periodically repeating sequence of subnormal numbers.
  (iii) Nan
  (iv) 10^m where m is roughly between -15 and -19.

Solution: (i). A similar experiment was done with the Fibonacci recurrence on Jan 10 in class and Homework 2, Problem 1, parts (e,f); due to roundoff you will numerically observe a solution that looks like $C_1(1 + \sqrt{2})^n + C_2(1 - \sqrt{2})^n$ **with**

$C_2 = 1$ but C_1 very small (but not exactly zero). Hence as $n \to \infty$, the $C_1(1+\sqrt{2})^n$ will be dominant; hence MATLAB reports $|x_n|$ as going beyond the limit of double precision. Since the recurrence for x_{n+2} involves only positive coefficients, x_n will be reported as Inf or -Inf for some large n (depending on the sign of C_1). Hence $|x_n|$ will be reported as Inf.

(d) What is the solution to this recurrence in the case $x_0 = 1$ and $x_1 = 1 + \sqrt{2}$? **Solution:** $x_n = (1 + \sqrt{2})^n$

(e) If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1 + \sqrt{2}$, for n large what will $|x_n|$ be reported as:
  (i) Inf
  (ii) 0 or some periodically repeating sequence of subnormal numbers.
  (iii) Nan
  (iv) 10^m where m is roughly between -15 and -19.
Solution: (i)

(8) Consider the recurrence

$$x_{n+2} = (8/7)x_{n+1} - (1/7)x_n.$$

(a) What is the general solution of this equation? Write this in the form $C_1 r_1^n + C_2 r_2^n$ for some $r_1, r_2 \in \mathbb{R}$.

(b) What is the solution to this recurrence in the case $x_0 = 1$ and $x_1 = 1/7$?

(c) This part is **multiple choice**: If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1/7$, for n large what will $|x_n|$ be reported as:
  (i) Inf
  (ii) 0 or some periodically repeating sequence of subnormal numbers.
  (iii) Nan
  (iv) 10^m where m is roughly between -15 and -19.
Solution: (iv); see Homework 1; since $r_1, r_2 = 1, 1/7$, the division by 7 has the same immediate roundoff/truncation error as in Problem 2 and 4 of Homework 1.

(9) Consider the recurrence

$$x_{n+2} = (9/8)x_{n+1} - (1/8)x_n.$$

(a) What is the general solution of this equation? Write this in the form $C_1 r_1^n + C_2 r_2^n$ for some $r_1, r_2 \in \mathbb{R}$. **Solution:** $r_1, r_2 = 1, 1/8$

(b) What is the solution to this recurrence in the case $x_0 = 1$ and $x_1 = 1/8$?

(c) This part is **multiple choice**: If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 1$ and $x_1 = 1/8$, for n large what will $|x_n|$ be reported as:
  (i) Inf
  (ii) 0 or some periodically repeating sequence of subnormal numbers.
  (iii) Nan
  (iv) 10^m where m is roughly between -15 and -19.

Solution: (ii). This is similar to Problems 1 and 3 on Homework 1.

(10) Consider the recurrence

$$x_{n+2} = x_{n+1} + 6x_n.$$

(a) What is the general solution of this equation? Write this in the form $C_1 r_1^n + C_2 r_2^n$ for some $r_1, r_2 \in \mathbb{R}$. **Solution:** $r_1, r_2 = 3, -2$

(b) What is the solution to this recurrence in the case $x_0 = 2$ and $x_1 = 1$? **Solution:** $x_0 = 2$ **implies** $C_1 + C_2 = 2$**, and** $x_1 = 1$ **implies** $3C_1 + (-2)C_2 = 1$**, which gives** $C_1 = C_2 = 1$**.**

(c) This part is **multiple choice**: If you numerically compute the values of $x_2, x_3, \ldots$ of this recurrence starting with $x_0 = 2$ and $x_1 = 1$, for n large what will $|x_n|$ be reported as:

(i) Inf
(ii) 0 or some periodically repeating sequence of subnormal numbers.
(iii) Nan
(iv) 10^m where m is roughly between -15 and -19.

Solution: (i)

(11) Find an exact formula for

$$\begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and justify your formula. [Hint: rather than diagonalize the above 2×2 matrix, it may be easier to relate this to a recurrence equation.] **Solution: This is related to the recurrence**

$$x_{n+2} = x_{n+1} + 6x_n;$$

since the solution to this recurrence with $x_1 = 1$ **and** $x_0 = 2$ **is** $x_n = 3^n + (-2)^n$**we have**

$$\begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3^{n+1} + (-2)^{n+1} \\ 3^n + (-2)^n \end{bmatrix}$$

(see Homework 2, all problems).

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