CPSC 303: FINAL PRACTICE QUESTIONS, SET 1, BRIEF SOLTIONS

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All questions on Homework 1–8 should be considered final exam practice.

RECALL THE FOLLOWING NOTATION AND DEFINITIONS REGARDING SPLINES AND Homework 7 and 8:

For fixed real numbers

$$A = x_0 < x_1 < \ldots < x_n = B,$$

and fixed $y_0, \ldots, y_n \in \mathbb{R}$, we set

$$\mathcal{U} = \mathcal{U}_{\mathbf{t},\mathbf{y}} \stackrel{\text{def}}{=} \Bigl\{ u \in C^2[A,B] \Bigm| u(x_i) = y_i \text{ for all } i \Bigr\}. \tag{1}$$

Also if $f\colon [A,B] \to \mathbb{R}$ is any function, we have set

$$\mathcal{U} = \mathcal{U}_{f;\mathbf{t}} \stackrel{\text{def}}{=} \Bigl\{ u \in C^2[A,B] \Bigm| u(x_i) = f(x_i) \text{ for all } i \Bigr\}, \tag{2}$$

and if f' exists at the endpoints x_0, x_n, then we considered the "clamped boundary" subspace of $\mathcal{U}_{f;\mathbf{t}}$ defined as

$$\Bigl\{ u \in \mathcal{U}_{f;\mathbf{t}} \Bigm| u'(x_0) = f'(x_0) \text{ and } u'(x_n) = f'(x_n) \Bigr\}.$$

For a cubic spline, $v(x)$, with endpoint x_0 and x_n and breakpoints $x_1 < \cdots < x_{n-1}$ we set

$$h_0 = x_1 - x_0,\ \ldots\ , h_{n-1} = x_n - x_{n-1},$$

and use the notation

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad \text{for } x_i \le x \le x_{i+1} \tag{3}$$

for the cubic pieces of $v(s)$.

The fundamental to compute cubic splines with either the free or clamped conditions involves the equations:

$$\frac{h_{i-1}}{h_{i-1} + h_i} c_{i-1} + 2c_i + \frac{h_i}{h_{i-1} + h_i} c_{i+1} = 3f[x_{i-1}, x_i, x_{i+1}], \quad i = 1, \ldots, n-1 \tag{4}$$

where $c_0 = c_n = 0$ for the free boundary conditions, and

$$2c_0 + c_1 = 3f[x_0, x_0, x_1] \quad \text{and} \quad c_{n-1} + 2c_n = 3f[x_{n-1}, x_{n-1}, x_n]$$

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under clamped boundary conditions.

In particular, if we consider the spline with all h_i equal, then our algorithm for the free boundary spline reduces to the equation

$$\big((1/2)N_{\mathrm{rod},n-1} + 2I\big)\mathbf{c} = 3\mathbf{\Phi}, \tag{5}$$

where $\mathbf{c} = (c_1, \ldots, c_{n-1})$, and where Φ is the $(n-1)$-dimensional vector whose i-th component equals $f[x_{i-1}, x_i, x_{i+1}]$ (and $c_0 = c_n = 0$). Hence we have

$$\mathbf{c} = (3/2)(I + (1/4)N_{\mathrm{rod}})^{-1}\mathbf{\Phi}.$$

Recall that σ refers to the operator on sequences $\{y_i\}_{i\in\mathbb{Z}}$ given by

$$(\sigma y)_i = y_{i+1},$$

and that we defined the difference operator $D = \sigma - 1$.

Recall that Homeworks 7 and 8 involved a number of matrices, including:

$$S_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_{n,-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

and

$$N_{\mathrm{rod},n} = S_{n,1} + S_{n,-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix};$$

and their variants

$$C_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{n,-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

and

$$N_{\mathrm{ring},n} = C_{n,1} + C_{n,-1}.$$

One can equivalently describe the above matrices as operators: i.e., these matrices are the unique matrices such that for all $(y_1, \ldots, y_n) \in \mathbb{R}^n$:

$$\begin{aligned}
S_{n,1}(y_1, \ldots, y_n) &= (y_2, \ldots, y_n, 0) \\
S_{n,-1}(y_1, \ldots, y_n) &= (0, y_1, \ldots, y_{n-1}) \\
C_{n,1}(y_1, \ldots, y_n) &= (y_2, \ldots, y_n, y_1) \\
C_{n,-1}(y_1, \ldots, y_n) &= (y_n, y_1, \ldots, y_{n-1})
\end{aligned}$$

For $k \in \mathbb{N} = \{1, 2, \ldots, \}$,

$$S_{n,k} = (S_{n,1})^k, \ S_{n,-k} = (S_{n,-1})^k, \ C_{n,k} = (C_{n,1})^k, \ C_{n,-k} = (C_{n,-1})^k.$$

We now recall our conventions regarding the heat equation in Homework 8; **you should be aware that outside CPSC 303 this term, the literature often has different notation and conventions**.

Recall that by the *heat equation* we mean the heat equation $u_t = u_{xx}$, i.e., $u_t(x,t) = u_{xx}(x,t)$ where (x,t) is a point in $\mathbb{R}^2$. [In the literature outside of CPSC 303 this year, there are more general heat equations, such as $u_t = (k(x)u_x)_x$ for a substance whose heat conductivity/capacity at x is reflected by $k(x)$; the case $k(x) = 1$ for all x is the above heat equation $u_t = u_{xx}$.]

We say that a function $u\colon [0,1] \times (0,\infty)$ is the solution to the *Dirichlet problem for the heat equation* we mean that $[0,1] \times [0,\infty)$ we mean

(1) $u_t(x,t) = u_{xx}(x,t)$ for all $(x,t) \in (0,1) \times (0,\infty)$ (i.e., all (x,t) with $0 < x < 1$ and all $t > 0$) (this is the heat equation); and
(2) $u(0,t) = u(1,t)$ for all $t > 0$ (in the literature outside of CPSC 303, this is sometimes called zero-valued Dirichlet condition; one can give more general *Dirichlet data* that specifies $u(0,t)$ and $u(1,t)$ which are two fixed, real constants, or even two functions of t.

Often we write $u(x,0) = f(x)$ for a function $f(x)$ that is given and is called the "initial condition" (i.e., the time $t = 0$ temperature profile of the rod). Sometimes we want u to be a continuous function on all of $[0,1] \times [0,\infty)$; if $f(x)$ above is continuous, this turns out to be equivalent to requiring that u be continuous at the two points $(0,0)$ and $(1,0)$.

We now recall our conventions regarding the discrete heat equation in Homework 8.

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$ and let $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ be the non-negative integers. We say that a function $U\colon \{0, 1, \ldots, n, n+1\} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfies the *discrete heat equation* if for all $i \in [n]$ and $j \in \mathbb{Z}_{\geq 0}$ we have

$$U(i, j+1) = U(i,j) + \theta\, D_i^{2,\mathrm{centre}} U(i,j), \tag{6}$$

where

$$D_i^{2,\mathrm{centre}} U(i,j) = U(i+1,j) + U(i-1,j) - 2U(i,j).$$

If $f\colon [n]\to\mathbb{R}$ is any function, we say that U satisfies the *initial condition* f, if

$$U(i,0)=f(i)\quad \text{for } i\in[n].$$

We say that U satisfies the *zero Dirichlet condition*, or simply the *Dirichlet condition*, if (6) holds for $i=1$ and $i=n$ (and all $j\ge 0$) provided that we have

$$U(0,j)=U(n+1,j)=0\quad \text{for all } j=1,2,\ldots.$$

The solution to the Dirichlet problem for the discrete heat equation can be written more simply as follows. If we use the notation

$$U(\cdot,j)=\begin{bmatrix}U(1,j)\\U(2,j)\\\vdots\\U(n,j)\end{bmatrix}$$

which we call the "temperature profile at time j," then one may write the solution to the Dirichlet problem for the discrete heat equation with initial value f as

$$U(\cdot,j)=\Bigl(I+\theta\bigl(N_{\mathrm{row},n}-2I\bigr)\Bigr)^j\mathbf{f}, \tag{7}$$

where

$$\mathbf{f}=\begin{bmatrix}f(1)\\f(2)\\\vdots\\f(n)\end{bmatrix}$$

is the intial value of U, i.e., $U(\cdot,0)=\mathbf{f}$. Equivalently, since $N_{\mathrm{row},n}=S_{n,1}+S_{n,-1}$, we can write

$$U(\cdot,j)=\Bigl(I+\theta\bigl(S_{n,1}+S_{n,-1}-2I\bigr)\Bigr)^j\mathbf{f},$$

(1) True/False

(a) The minimizer of

$$\mathcal{E}(u) = \int_A^B \big(u''(x)\big)^2 \, dx$$

over $\mathcal{U}_{f;\mathbf{t}}$ is unique.

(b) The minimizer, v, of the energy

$$\mathcal{E}(u) = \int_A^B \big(u''(x)\big)^2 \, dx$$

over $\mathcal{U}_{f;\mathbf{t}}$ satisfies $v''(t_0) = v''(t_n) = 0$.

(c) The minimizer, v, of the energy

$$\mathcal{E}(u) = \int_A^B \big(u''(x)\big)^2 \, dx$$

over $\mathcal{U}_{f;\mathbf{t}}$ corresponds to the "free boundary" condition.

(d) The minimizer, v, of the energy

$$\mathcal{E}(u) = \int_A^B \big(u''(x)\big)^2 \, dx$$

over $\mathcal{U}_{f;\mathbf{t}}$ corresponds to the "clamped boundary" condition of f.

(e) If $h_0 = \cdots = h_n$, then the equations for the $\mathbf{c} = (c_1, \ldots, c_{n-1})$ under the free boundary condition correspond to

$$\big((1/2)N_{\mathrm{rod},n-1} + 2I\big)\mathbf{c} = 3\mathbf{F},$$

where $\mathbf{F}$ is the vector whose ith component is $f[x_{i-1}, x_i, x_{i+1}]$.

(f) For $n \geq 1$, the inverse of $S_{n,1}$ is $S_{n,-1}$.

(g) For $n \geq 1$, the inverse of $C_{n,1}$ is $C_{n,-1}$.

(h) The function $u(x,t) = e^x \sin(x)$ satisfies the heat equation $u_t = u_{xx}$ throughout $\mathbb{R}^2$ (i.e., $u_t(x,t) = u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R}^2$).

The following corrections were made at 3:18pm on April 15.

(i) The function $u(x,t) = e^t \sin(x)$ satisfies the heat equation $u_t = u_{xx}$ throughout $\mathbb{R}^2$ (i.e., $u_t(x,t) = u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R}^2$).

(j) The function $u(x,t) = e^{-t} \sin(x)$ satisfies the heat equation $u_t = u_{xx}$ throughout $\mathbb{R}^2$ (i.e., $u_t(x,t) = u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R}^2$).

(k) For any $\omega \in \mathbb{R}$, the function $u(x,t) = e^{-\omega^2 t} \sin(\omega x)$ satisfies the heat equation throughout $\mathbb{R}^2$.

(l) For any $\omega \in \mathbb{R}$, the function $u(x,t) = e^{\omega^2 t} \sin(\omega x)$ satisfies the heat equation throughout $\mathbb{R}^2$.

(m) For any $\omega \in \mathbb{R}$, the function $u(x,t) = e^{\omega^2 t} \sin(\omega x)$ satisfies the equation $-u_t = u_{xx}$throughout $\mathbb{R}^2$.

(n) One solution to the Dirichlet problem for the heat equation $[0,1] \times (0,\infty)$ is the function $u(x,t) = \sin(x)e^{-t}$.

(2) **Circle the correct numeral (i, ii, iii, or iv) in each of the following questions.**

(a) Interpolating a fuction, $f = f(x)$, at a large number of values $x_0, \ldots, x_n$ data points is disadvantageous because:
 (i) the error in interpolation formula may not be small if the $(n+1)$-st derivative of f is very large (or doesn't exist);
 (ii) adding a single data point can drastically change the entire interpolant;
 (iii) the values of the interpolant at any point can greatly depend on far away values of f;
 (iv) all of the above.

(b) For $n \geq 2$, $\|S_{n,1}\|_\infty$
 (i) equals 1 for all $n \geq 2$;
 (ii) is at most 1 but not always equal to 1;
 (iii) equals 2;
 (iv) does not exist.

(c) All $n \in \mathbb{N}$, $S_{n,1}(y_1, \ldots, y_n)$ equals
 (i) $(y_2, y_3, \ldots, y_n, 0)$;
 (ii) $(y_2, y_3, \ldots, y_n, y_1)$;
 (iii) $(0, y_1, \ldots, y_{n-2}, y_{n-1})$;
 (iv) $(y_n, y_1, \ldots, y_{n-2}, y_{n-1})$.

(d) Fix $n \in \mathbb{N}$. The set of k for which $S_{n,1}^k = 0$ is
 (i) $\emptyset$ (the empty set);
 (ii) $k \geq 2$;
 (iii) $k \geq n$;
 (iv) $k \geq n + 1$.

(e) Fix $n \in \mathbb{N}$. The set of k for which $S_{n,-1}^k = 0$ is
 (i) $\emptyset$ (the empty set);
 (ii) $k \geq 2$;
 (iii) $k \geq n$;
 (iv) $k \geq n + 1$.

(f) Fix $n \in \mathbb{N}$. The set of k for which $C_{n,1}^k = 0$ is
 (i) $\emptyset$ (the empty set);
 (ii) $k \geq 2$;
 (iii) $k \geq n$;
 (iv) $k \geq n + 1$.

(g) Fix $n \in \mathbb{N}$. The set of k for which $C_{n,-1}^k = 0$ is
 (i) $\emptyset$ (the empty set);
 (ii) $k \geq 2$;
 (iii) $k \geq n$;
 (iv) $k \geq n + 1$.

(h) Consider the discrete heat equation with $n = 1$ houses, $\theta = -1/3$ and $f(1) = 4$. As $j \to \infty$, $U(1, j)$,
 (i) is always positive and tends to infinity;
 (ii) alternates in sign between positive and negative and its absolute value tends to infinity;
 (iii) is always positive and tends to zero;

   (iv) alternates in sign between positive and negative and its absolute value tends to zero.

(i) Consider the discrete heat equation with $n = 1$ houses, $\theta = 1/3$ and $f(1) = 4$. As $j \to \infty$, $U(1, j)$,
   (i) is always positive and tends to infinity;
   (ii) alternates in sign between positive and negative and its absolute value tends to infinity;
   (iii) is always positive and tends to zero;
   (iv) alternates in sign between positive and negative and its absolute value tends to zero.

(j) Consider the discrete heat equation with $n = 1$ houses, $\theta = 1$ and $f(1) = 4$. As $j \to \infty$, $U(1, j)$,
   (i) alternates in sign between positive and negative and its absolute value tends to infinity;
   (ii) alternates in sign between positive and negative and its absolute value always equals 4;
   (iii) is always positive and tends to zero;
   (iv) alternates in sign between positive and negative and its absolute value tends to zero.

(k) Consider the discrete heat equation with $n = 1$ houses, $\theta = 2$ and $f(1) = 4$. As $j \to \infty$, $U(1, j)$,
   (i) alternates in sign between positive and negative and its absolute value tends to infinity;
   (ii) alternates in sign between positive and negative and its absolute value always equals 4;
   (iii) is always positive and tends to zero;
   (iv) alternates in sign between positive and negative and its absolute value tends to zero.

(l) Consider the discrete heat equation with $n = 2$ houses, $\theta = 1/4$ and any $\mathbf{f} = (5, 6)$. As $j \to \infty$, $U(1, j)$ and $U(2, j)$
   (i) both alternate in sign between positive and negative, and each of their absolute values tends to infinity;
   (ii) both alternate in sign between positive and negative and both of their absolute values tend to zero;
   (iii) both are always positive and tends to zero;
   (iv) both are always positive and tend to infinity.

(3) Write short answers. For example, if the answer is 1.5 or 3/2, either form is acceptable. We will do our best to accept some forms that are not fully reduced: for example, if a formula produces 6/4, then that's OK, too; it is not OK to unecessarily introduce a factor of 13524 in the numerator and denominator and write the answer as 81144/54096.

(a) For $n = 2$, what is $\|N_{\mathrm{rod},n}\|_\infty$?

(b) For $n = 3$, what is $\|N_{\mathrm{rod},n}\|_\infty$?

(c) Consider the discrete heat equation with $n = 2$ houses (Homework 8, Section 4), and $\theta = 1/3$. If initially House 1 is at 5°C and House 2 at 1°C, what is the temperature of House 1 at time $j = 1$ and $j = 2$?

(d) Let A be an $n \times n$ matrix with $\|A\|_\infty \le 1/2$, and let

$$U = U(A) = (I - A)^{-1} - (I - A + A^2 - A^3).$$

Give the best possible upper bound on $\|U\|_\infty$, i.e., find an $M \in \mathbb{R}$ such that $\|U\|_\infty \le M$ for all A (with $\|A\|_\infty \le 1/2$), and give an A such that $\|U\|_\infty = M$.

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