CPSC 303 HOMEWORK 8, SOME SOLUTIONS, SPRING 2020

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1. Section 1

Exercise 1.1. Which of the following functions satisfy the heat equation $u_t(x,t) = u_{xx}(x,t)$ for all $x, t \in \mathbb{R}$?

1.1(a) $u(x,t) = 3 + 4x$.
1.1(b) $u(x,t) = 3 + 4x + x^2$.
1.1(c) $u(x,t) = 3 + 4x + x^2 - t$.

Solution: $u_x = 4 + 2x$, $u_{xx} = (4 + 2x)_x = 2$, and $u_t = -1$. So no, it is not true that $u_t(x,t) = u_{xx}(x,t)$ for all $x, t \in \mathbb{R}$.

1.1(d) $u(x,t) = 3 + 4x + x^2 - 2t$.
1.1(e) $u(x,t) = \sin(\omega x)e^{-t\omega^2}$ for any constant $\omega \in \mathbb{R}$.

Solution: $u_x = \omega \cos(\omega x)e^{-t\omega^2}$, $u_{xx} = -\omega^2 \sin(\omega x)e^{-t\omega^2}$, $u_t = -\omega^2 \sin(\omega x)e^{-t\omega^2}$. Hence u satisfies $u_{xx} = u_t$ throughout $\mathbb{R}^2$.

1.1(f) $u(x,t) = \cos(\omega x)e^{-t\omega^2}$ for any constant $\omega \in \mathbb{R}$.

Solution: $u_{xx} = u_t = -\omega^2 \cos(\omega x)e^{-t\omega^2}$.

1.1(g) $u(x,t) = \sinh(\omega x)e^{t\omega^2}$ for any constant $\omega \in \mathbb{R}$ (where $\sinh(x) \stackrel{\text{def}}{=} (e^x - e^{-x})/2$) is the *hyperbolic sine*.

Solution: $u_{xx} = u_t = \omega^2 \sinh(\omega x)e^{t\omega^2}$.

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Exercise 1.2. For which $(x,t) \in \mathbb{R}^2$ is it true that $u_t(x,t) = u_{xx}(x,t)$ for the function:

1.2(a) $u(x,t) = 4x + t^2$?

Solution: $u_x = 4$, $u_{xx} = 0$, and $u_t = 2t$. Hence $u_t = u_{xx}$ whenever $2t = 0$, i.e., when $t = 0$ (and x is arbitrary).

1.2(b) $u(x,t) = 4x^3 + t^2$?

Solution: $u_x = 12x^2$, $u_{xx} = 24x$, and $u_t = 2t$. Hence $u_t = u_{xx}$ whenever $2t = 24x$, i.e., whenever $t = 12x$.

1.2(c) $u(x,t) = 2x^2 + 3t^3$?

Solution: $u_x = 4x$, $u_{xx} = 4$, and $u_t = 9t^2$. Hence $u_t = u_{xx}$ whenever $9t^2 = 4$, when $t^2 = 4/9$, i.e., when $t = \pm 2/3$ (and x is arbitrary).

1.2(d) $u(x,t) = 2x^2 - 3t^3$?

Exercise 1.3. Show that if $u(x,t), v(x,t)$ both satisfy the heat equation at some point $(x_0,t_0) \in \mathbb{R}^2$, and $\alpha, \beta \in \mathbb{R}$, then

$$w = w(x,t) \stackrel{\text{def}}{=} \alpha u(x,t) + \beta v(x,t)$$

also satisfies the heat equation at (x_0,t_0).

Solution: We have

$$w_x = \bigl(\alpha u + \beta v\bigr)_x = \alpha u_x + \beta v_x.$$

Similarly

$$w_{xx} = \bigl(\alpha u_x + \beta v_x\bigr)_x = \alpha u_{xx} + \beta v_{xx}.$$

And similarly

$$w_t = \bigl(\alpha u + \beta v\bigr)_t = \alpha u_t + \beta v_t.$$

So if u, v both satisfy the heat equation at (x_0,t_0) then

$$w_t(x_0,t_0) = \alpha u_t(x_0,t_0) + \beta v_t(x_0,t_0) = \alpha u_{xx}(x_0,t_0) + \beta v_{xx}(x_0,t_0) = w_x(x_0,t_0).$$

Hence w satisfies the heat equation at (x_0,t_0).

Exercise 1.4. Let $R \subset \mathbb{R}^2$ be a subset ("R" is to suggest the word "region"). Let $c_0, c_1 \in \mathbb{R}$, and $u\colon \mathbb{R}^2 \to \mathbb{R}$ be any function, and let

$$v(x,t) = u(x,t) - (c_0 + c_1 x).$$

1.4(a) Show that u satisfies the heat equation in R, i.e., $u_t(x,t) = u_{xx}(x,t)$, for all $(x,t) \in R$, iff v satisfies the heat equation in R.

Solution: We have $v_x = u_x - c_1$ and hence $v_{xx} = u_{xx}$. Also $v_t = u_t - (c_0 + c_1 x)_t = u_t$. Hence $u_t = v_t$ and $u_{xx} = v_{xx}$. It follows that for any fixed (x,t), $u_t(x,t) = u_{xx}(x,t)$ iff $v_t(x,t) = v_{xx}(x,t)$. Hence u satisfies the heat equation for all $(x,t) \in R$ iff v does.

1.4(b) Show that for any $a, b \in \mathbb{R}$ there is a unique c_0, c_1 such that $c_0 = a$ and $c_0 + c_1 = b$.

Solution: Subtracting the first equation from the second we have that $c_0 = a$ and $c_0 + c_1 = b$ is equivalent to the system $c_0 = a$ and $c_1 == b - a$; this determines c_0, c_1 uniquely in terms of a, b.

1.4(c) Show that if we replace the Dirichlet problem by more general Dirichlet data:

$$u(0,t) = a, \quad u(1,t) = b$$

for some constants $a, b \in \mathbb{R}$, it is equivalent to substitute

$$v(x,t) = u(x,t) - (c_0 + c_1 x)$$

(with c_0, c_1 as above) and solve the (zero-valued) Dirichlet problem above, replacing $f(x)$ with $f(x) - c_0 - c_1 x$. [Remark: for simplicity we took $u\colon \mathbb{R}^2 \to \mathbb{R}$ instead of $u\colon [0,1] \times [0,\infty) \to \mathbb{R}$; should this bother us?]

Solution: First, from part (a), u satisfies the heat equation in $(0,1) \times (0,\infty)$ iff v does. Second, $u(0,t) = a$ for all $t > 0$ iff $v(0,t) = u(0,t) - c_0 = a - a = 0$ (for all $t > 0$). Third, $u(1,t) = b$ for all $t > 0$ iff $v(1,t) = u(1,t) - (c_0 + c_1) = b - b = 0$ (for all $t > 0$). Fourth, $u(0,x) = f(x)$ iff $v(0,x) = u(0,x) - (c_0 + c_1 x) = f(x) - c_0 - c_1 x$. This implies the equivalence of the above Dirichlet problem for v with the (not necessarily zero-valued) Dirichlet problem for u.

Note that the above equivalence uses only the values of u and v on $[0,1] \times [0,\infty)$, so u, v need only be defined on $[0,1] \times [0,\infty)$.

3. Section 3

Exercise 3.1. Show that

$$D^{2,\mathrm{centre}} = (\sigma - 1)(1 - \sigma^{-1}) = \sigma + \sigma^{-1} - 2.$$

Solution: By definition

$$D^{2,\mathrm{centre}} = \sigma + \sigma^{-1} - 2.$$

But

$$(\sigma - 1)(1 - \sigma^{-1}) = \sigma(1 - \sigma^{-1}) - 1(1 - \sigma^{-1}) = \sigma - \sigma\sigma^{-1} - 1 + \sigma^{-1} = \sigma + \sigma^{-1} - 2.$$

Exercise 3.2. Show that

$$D^{2,\mathrm{centre}} = \sigma^{-1} D^2.$$

Solution:

$$\sigma^{-1} D^2 = \sigma^{-1}(\sigma - 1)^2 = \sigma^{-1}\big(\sigma^2 - 2\sigma + 1\big) = \sigma - 2 + \sigma^{-1}$$

which, by definition, equals $D^{2,\mathrm{centre}}$.

Exercise 3.3. Say that instead of considering sequences

$$\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}} = \{\ldots, y_{-1}, y_0, y_1, y_2, \ldots\}$$

we consider only the "truncated" finite sequences

$$\mathbf{y} = (y_1, \ldots, y_n)$$

(you might think of the doubly infinite sequence where we enforce that $y_i = 0$ for all $i > n$ and all $i < 1$). Say that we define σ_{trunc} to be the operator

$$\sigma_{\mathrm{trunc}}(y_1, \ldots, y_n) \stackrel{\mathrm{def}}{=} (y_2, y_3, \ldots, y_n, 0).$$

Show that in matrix form, we have

$$\sigma_{\mathrm{trunc}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = S_{n,1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

where $S_{n,1}$ is the matrix given in Homework 7 (and in the appendices here).

Solution: We have

$$\sigma_{\mathrm{trunc}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ 0 \end{bmatrix}$$

by definition. Also

$$S_{n,1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ 0 \end{bmatrix}$$

Comparing the right-hand-sides of the two equations displayed above gives the desired equality.

4. Section 4

Exercise 4.1. Consider the discrete heat equation in the case $n = 1$; therefore there is only House 1, and it is incident upon a fictitious House 0 and House 2 that are always at 0°C.

4.1(a) Show that for any $j \geq 0$, we have

$$\mathrm{Temp}(1, j+1) = (1 - 2\theta)\mathrm{Temp}(j, 1).$$

Solution: By definition we have that

$$\mathrm{Temp}(1, j+1) = \mathrm{Temp}(1, j) + \theta\, D_i^{2,\mathrm{centre}} \mathrm{Temp}(i, j)$$
$$= \mathrm{Temp}(1, j) + \theta\bigl(\mathrm{Temp}(0, j) + \mathrm{Temp}(2, j) - 2\mathrm{Temp}(1, j)\bigr) = \mathrm{Temp}(1, j)\bigl(1 - 2\theta\bigr).$$

4.1(b) Given the initial temperature $\mathrm{Temp}(1, 0) = f(1)$ of House 1, describe $\mathrm{Temp}(1, j)$ as a function of $f(1)$ and j.

Solution:

$$\mathrm{Temp}(1, j) = (1 - 2\theta)\mathrm{Temp}(1, j-1)$$
$$= (1 - 2\theta)^2\mathrm{Temp}(1, j-2) = \cdots = (1 - 2\theta)^j\mathrm{Temp}(1, 0) = (1 - 2\theta)^j f(1)$$

4.1(c) Show that for $0 < \theta \leq 1/2$, and $f(1) > 0$, $\mathrm{Temp}(1, j)$ is positive for all j and $\mathrm{Temp}(1, j) \to 0$ as $j \to \infty$.

Solution: In this case

$$0 > -2\theta > -1$$

and therefore

$$1 > 1 - 2\theta > 0.$$

Hence $(1 - 2\theta)^j$ is positive for all $j > 0$, and $(1 - 2\theta)^j \to 0$ as $j \to \infty$. If $f(1) > 0$ then $(1 - 2\theta)^j f(1)$ is positive for all $j > 0$, and $(1 - 2\theta)^j f(1) \to 0$ as $j \to \infty$.

4.1(d) If $1/2 < \theta < 1$ and $f(1) > 0$, how does the sign of $\mathrm{Temp}(1, j)$ behave?

Solution: In this case

$$-1 > -2\theta > -2$$

and therefore

$$0 > 1 - 2\theta > -1.$$

It follows that $(1 - 2\theta)^j$ is positive for j even and negative for j odd. If $f(1) > 0$, then the same is true for $(1 - 2\theta)^j f(1)$.

4.1(e) If $\theta < 0$ and $f(1) > 0$, what happens to $\mathrm{Temp}(1, j)$ as $j \to \infty$?

Solution: if $\theta < 0$ then $-2\theta > 0$ and hence $1 - 2\theta > 1$. Hence $(1 - 2\theta)^j \to \infty$ as $j \to \infty$. If $f(1) > 0$, then the same is true for $(1 - 2\theta)^j f(1)$.

Exercise 4.2. Consider the case of $n = 1$ in Exercise 4.1, with $\mathrm{Temp}(1, 0) = f(1)$ a fixed value. Fix an integer $T > 0$. For any $m \in \mathbb{N}$, let $\theta = 1/m$ and set $g(m) = \mathrm{Temp}(1, Tm)$ (with $\theta = 1/m$). Show that

$$\lim_{m\to\infty} g(m) = e^{-2\theta T} f(1).$$

[Hint: first show that $g(m) = (1 - 2/m)^{Tm}$. Then apply ln to both sides; at this point you can use l'Hôpital's rule or Taylor's theorem.]

Solution: According to Exercise 4.1,

$$g(m) = (1 - 2\theta)^{Tm} f(1) = (1 - 2/m)^{Tm} f(1).$$

So first consider

$$L = \lim_{m\to\infty} (1 - 2/m)^m.$$

Applying ln to both sides we have

$$\ln(L) = \lim_{m\to\infty} m \ln(1 - 2/m).$$

Taylor's theorem implies that as $m \to \infty$,

$$\ln\bigl(1 - 2/m\bigr) = -2/m + O(1/m^2),$$

and hence

$$\ln(L) = \lim_{m\to\infty} Tm\bigl(-2/m + O(1/m^2)\bigr) = -2T.$$

Hence

$$L = e^{-2T}$$

and so

$$\lim_{m\to\infty} g(m) = f(1) \lim_{m\to\infty} (1 - 2/m)^{Tm} = f(1)e^{-2T}.$$

Exercise 4.3. Let $n = 2$ and $\theta \in \mathbb{R}$ be arbitrary.

4.3(a) Show that if for some j we have $\mathrm{Temp}(1, j) = \mathrm{Temp}(2, j)$, then

$$\mathrm{Temp}(1, j + 1) = \mathrm{Temp}(2, j + 1) = (1 - \theta)\mathrm{Temp}(1, j) = (1 - \theta)\mathrm{Temp}(2, j)$$

Solution: We have

$$\mathrm{Temp}(1, j + 1) = \mathrm{Temp}(1, j) + \theta\, D_i^{2,\mathrm{centre}}\mathrm{Temp}(i, j)$$
$$= \mathrm{Temp}(1, j) + \theta\bigl(\mathrm{Temp}(0, j) + \mathrm{Temp}(2, j) - 2\mathrm{Temp}(1, j)\bigr)$$
$$= \mathrm{Temp}(1, j) + \theta\bigl(0 + \mathrm{Temp}(1, j) - 2\mathrm{Temp}(1, j)\bigr) = \mathrm{Temp}(1, j)(1 - \theta)$$

Similarly

$$\mathrm{Temp}(2, j + 1) = \mathrm{Temp}(2, j) + \theta\bigl(\mathrm{Temp}(1, j) + \mathrm{Temp}(3, j) - 2\mathrm{Temp}(2, j)\bigr)$$
$$= \mathrm{Temp}(2, j) + \theta\bigl(\mathrm{Temp}(2, j) + 0 - 2\mathrm{Temp}(2, j)\bigr) = \mathrm{Temp}(2, j)(1 - \theta).$$

4.3(b) Assume that $f\colon \{1,2\} \to \mathbb{R}$ satisfies $f(1) = f(2)$, and let $\mathrm{Temp}(i, 0) = f(i)$ for $i = 1, 2$. Show that
$$\mathrm{Temp}(1, j) = \mathrm{Temp}(2, j) = (1-\theta)^j f(1).$$

Solution: By the previous part we have
$$\mathrm{Temp}(1, 1) = (1-\theta)f(1) = (1-\theta)f(2) = \mathrm{Temp}(2, 1).$$
Simarly
$$\mathrm{Temp}(1, 2) = \mathrm{Temp}(2, 2) = (1-\theta)^2 f(1).$$
Applying this argument repeatedly (or, by induction on j)
$$\mathrm{Temp}(1, j) = \mathrm{Temp}(2, j) = (1-\theta)^j f(1).$$

4.3(c) Say that $f(1) = f(2) > 0$, and that $0 < \theta < 1$. What is
$$\lim_{j\to\infty} \mathrm{Temp}(1, j)$$
(and justify your answer)?

Solution: In this case $0 > -\theta > -1$ and so $1 > 1-\theta > 0$. Hence $(1-\theta)^j \to 0$ as $j \to \infty$, and hence
$$\lim_{j\to\infty} \mathrm{Temp}(1, j) = \lim_{j\to\infty} (1-\theta)^j f(1) = 0.$$

4.3(d) Say that $f(1) = f(2) > 0$, and that $\theta < 0$. What is
$$\lim_{j\to\infty} \mathrm{Temp}(1, j)$$
(and justify your answer)?

Solution: In this case $0 < -\theta$ and so $1 < 1-\theta$. Hence $(1-\theta)^j \to \infty$ as $j \to \infty$, and hence
$$\lim_{j\to\infty} \mathrm{Temp}(1, j) = \lim_{j\to\infty} (1-\theta)^j f(1) = \infty$$

5. Section 5

Exercise 5.1. Say that for all j we gather all the temperatures at time j into a vector,

$$\mathrm{TempProfile}(j) = \begin{bmatrix} \mathrm{Temp}(1,j) \\ \mathrm{Temp}(2,j) \\ \vdots \\ \mathrm{Temp}(n,j) \end{bmatrix} \tag{1}$$

5.1(a) Show that the discrete heat equation implies that for all $j \geq 0$ we have

$$\mathrm{TempProfile}(j+1) = \Big(I + \theta\big(N_{\mathrm{rod},n} - 2I\big)\Big)\mathrm{TempProfile}(j),$$

where $N_{\mathrm{rod},n}$ is the matrix given in Appendix **??**.

Solution: For each $j > 0$ and each i between 2 and n we have

$$\mathrm{Temp}(i,j+1) = \mathrm{Temp}(i,j) + \theta\big(\mathrm{Temp}(i-1,j) + \mathrm{Temp}(i+1,j) - 2\mathrm{Temp}(i,j)\big),$$

and the same is true of $i = 1$ and $i = n$ provided we ignore $\mathrm{Temp}(0,j) = 0 = \mathrm{Temp}(n+1,j)$. Hence

$$\begin{bmatrix} \mathrm{Temp}(1,j+1) \\ \mathrm{Temp}(2,j+1) \\ \vdots \\ \mathrm{Temp}(n-1,j+1) \\ \mathrm{Temp}(n,j+1) \end{bmatrix} = \begin{bmatrix} \mathrm{Temp}(1,j) \\ \mathrm{Temp}(2,j) \\ \vdots \\ \mathrm{Temp}(n-1,j) \\ \mathrm{Temp}(n,j) \end{bmatrix} + \theta\left(\begin{bmatrix} 0 \\ \mathrm{Temp}(1,j) \\ \vdots \\ \mathrm{Temp}(n-2,j) \\ \mathrm{Temp}(n-1,j) \end{bmatrix} + \begin{bmatrix} \mathrm{Temp}(2,j) \\ \mathrm{Temp}(3,j) \\ \vdots \\ \mathrm{Temp}(n,j) \\ 0 \end{bmatrix} - 2 \begin{bmatrix} \mathrm{Temp}(1,j) \\ \mathrm{Temp}(2,j) \\ \vdots \\ \mathrm{Temp}(n-1,j) \\ \mathrm{Temp}(n,j) \end{bmatrix} \right)$$

Now we recognize

$$\begin{bmatrix} \mathrm{Temp}(2,j) \\ \mathrm{Temp}(3,j) \\ \vdots \\ \mathrm{Temp}(n,j) \\ 0 \end{bmatrix} = S_{n,1} \begin{bmatrix} \mathrm{Temp}(1,j) \\ \mathrm{Temp}(2,j) \\ \vdots \\ \mathrm{Temp}(n-1,j) \\ \mathrm{Temp}(n,j) \end{bmatrix}$$

and similarly

$$\begin{bmatrix} 0 \\ \mathrm{Temp}(1,j) \\ \vdots \\ \mathrm{Temp}(n-2,j) \\ \mathrm{Temp}(n-1,j) \end{bmatrix} = S_{n,-1} \begin{bmatrix} \mathrm{Temp}(1,j) \\ \mathrm{Temp}(2,j) \\ \vdots \\ \mathrm{Temp}(n-1,j) \\ \mathrm{Temp}(n,j) \end{bmatrix}$$

where $S_{n,1}$ and $S_{n,-1}$ are the shift matrices/operators given in Appendix B. Hence

$$\begin{aligned} \mathrm{TempProfile}(j+1) &= \mathrm{TempProfile}(j) + \theta\big(S_{n,-1} + S_{n,1} - 2I\big)\mathrm{TempProfile}(j) \\ &= \Big(I + \theta\big(N_{\mathrm{rod},n} - 2I\big)\Big)\mathrm{TempProfile}(j). \end{aligned}$$

5.1(b) Show that for any $j \geq 0$ we have

$$\mathrm{TempProfile}(j) = \Bigl(I + \theta\bigl(N_{\mathrm{rod},n} - 2I\bigr)\Bigr)^j \mathrm{TempProfile}(0).$$

Solution:

$$\mathrm{TempProfile}(j) = \Bigl(I + \theta\bigl(N_{\mathrm{rod},n} - 2I\bigr)\Bigr)\mathrm{TempProfile}(j-1)$$

$$= \Bigl(I+\theta\bigl(N_{\mathrm{rod},n}-2I\bigr)\Bigr)^2 \mathrm{TempProfile}(j-2) = \cdots = \Bigl(I+\theta\bigl(N_{\mathrm{rod},n}-2I\bigr)\Bigr)^j \mathrm{TempProfile}(0).$$

Exercise 5.2. Let $n = 2$ and $\theta = 1/3$, and let TempProfile be as in (1).

5.2(a) Show that for any $j \geq 0$

$$\mathrm{TempProfile}(j) = \left(\frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)^j \mathrm{TempProfile}(0)$$

Solution: We have

$$\begin{aligned} I + \theta\bigl(N_{\mathrm{rod},n} - 2I\bigr) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + (1/3)\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \end{aligned}$$

The result now follows from Exercise 5.1(b).

5.2(b) Find a formula for $\mathrm{TempProfile}(j)$ for $j \geq 1$ in terms of $a = \mathrm{Temp}(1,0) + \mathrm{Temp}(2,0)$.

Solution:

$$\mathrm{TempProfile}(1) = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\mathrm{TempProfile}(0) = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\mathrm{Temp}(1,0)\\ \mathrm{Temp}(2,0)\end{bmatrix} = \begin{bmatrix}a/3\\ a/3\end{bmatrix}$$

Then

$$\mathrm{TempProfile}(2) = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}a/3\\ a/3\end{bmatrix} = \frac{2}{3}\begin{bmatrix}a/3\\ a/3\end{bmatrix}.$$

Similarly

$$\mathrm{TempProfile}(3) = \left(\frac{2}{3}\right)^2\begin{bmatrix}a/3\\ a/3\end{bmatrix},$$

and repeating this arguments shows that for $j \geq 1$,

$$\mathrm{TempProfile}(j) = \left(\frac{2}{3}\right)^{j-1}\begin{bmatrix}a/3\\ a/3\end{bmatrix}.$$

6. Section 6

Exercise 6.1. Imagine a variant of the n-House discrete heat transfer equation, where the left side of House 1 is connected to the right side of House n, instead of each connected to a 0°C heat absorbing sink[1]. In this case the discrete heat equation is the same, except that we identify House 0 with House n and House $n+1$ with House 1, i.e., we work with house numbers "modulo n." More precisely we define

$$\mathrm{Temp}_{\mathrm{ring}}(i,j) = \left\{ \begin{array}{ll} \mathrm{Temp}(n,j) & \text{if } i=0, \\ \mathrm{Temp}(i,j) & \text{if } 1 \le i \le n-1, \text{ and} \\ \mathrm{Temp}(1,j) & \text{if } i = n+1. \end{array} \right\}$$

and (10) becomes

$$\mathrm{Temp}_{\mathrm{ring}}(i,j+1) = \mathrm{Temp}_{\mathrm{ring}}(i,j) + \theta\, D_i^{2,\mathrm{centre}} \mathrm{Temp}_{\mathrm{ring}}(i,j).$$

Show that with notation as in (13), we have

$$\mathrm{TempProfile}(j) = \Big(I + \theta \big(N_{\mathrm{ring},n} - 2I \big) \Big)^j \mathrm{TempProfile}(0),$$

where $N_{\mathrm{ring},n}$ is the matrix given in Appendix B.

Solution: Because for the modulo n definition of $\mathrm{Temp}_{\mathrm{ring}}$, we get the same result as in Exercise 5.1, except that instead of

$$\begin{bmatrix} 0 \\ \mathrm{Temp}(1,j) \\ \vdots \\ \mathrm{Temp}(n-2,j) \\ \mathrm{Temp}(n-1,j) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathrm{Temp}(2,j) \\ \mathrm{Temp}(3,j) \\ \vdots \\ \mathrm{Temp}(n,j) \\ 0 \end{bmatrix},$$

we get

$$\begin{bmatrix} \mathrm{Temp}(n,j) \\ \mathrm{Temp}(1,j) \\ \vdots \\ \mathrm{Temp}(n-2,j) \\ \mathrm{Temp}(n-1,j) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathrm{Temp}(2,j) \\ \mathrm{Temp}(3,j) \\ \vdots \\ \mathrm{Temp}(n,j) \\ \mathrm{Temp}(1,j) \end{bmatrix}.$$

Since the 0's are now replaced with $\mathrm{Temp}(n,j)$ and $\mathrm{Temp}(1,j)$, which gives $N_{\mathrm{rod},n}$ with 1's in the upper right and lower left corners of the matrix, so $N_{\mathrm{ring},n}$ replaces $N_{\mathrm{rod},n}$ in the solution to Exercise 5.1.

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[1] Perhaps n is very large and the houses are in a large "ring" in 2-dimensions, which is more simply modeled by a thin, 1-dimensional loop. Or maybe what seems like a flat, infinite 1-dimensional universe actually "wraps" around itself. Feel free to choose the story here.