CPSC 303 HOMEWORK 7 SOLUTIONS, SPRING 2020

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(1) Let us make some calculations regarding the matrices $S_{n,1}$ and $S_{n,-1}$ in the case $n = 3$, where

$$S_{3,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_{3,-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Using regular matrix multiplication, compute the expressions $S_{3,1}S_{3,-1}$ and $S_{3,-1}S_{3,1}$. Are they equal?

(b) Compute the way $S_{3,1}S_{3,-1}$ operates on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ as

$$S_{3,1}\bigl(S_{3,-1}\mathbf{x}\bigr),$$

i.e., by first computing

$$S_{3,-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and then applying $S_{3,1}$ to the result.

(c) Similarly compute the way that $S_{3,-1}S_{3,1}$ operates on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

(d) Using matrix multiplications, compute the values of

$$S_{3,1}^2, \quad S_{3,1}^3.$$

(e) Compute the way that

$$S_{3,1}^2, \quad S_{3,1}^3$$

operate on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ by applying twice and three times, respectively, the operation of $S_{3,1}$ on $\mathbf{x}$.

(f) Compute the value of

$$e^{S_{3,1}t}$$

using the formula

$$e^{At} = I + (At) + (At)^2/2 + (At)^3/3! + \cdots$$

using either direct matrix calculations or operator calculations as above.

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(g) Compute the value of $(I+S_{3,1})^{-1}$ using (5) (of the Homework 7 handout). Check your work by multiplying this result by $I+S_{3,1}$ to see that you get the identity matrix.

Solution:

(a)

$$S_{3,1}S_{3,-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$S_{3,-1}S_{3,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$S_{3,1}\bigl(S_{3,-1}\mathbf{x}\bigr) = S_{3,1}\left(S_{3,-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

which using the formula for $S_{n,-1}\mathbf{x}$ in Section 2.1.

$$= S_{3,1} \begin{bmatrix} 0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

by the formula for $S_{n,1}\mathbf{x}$ in Section 2.1 (of the material in Homework 7).

(c) Similarly, by the formulas in Section 2.1 we have

$$S_{3,-1}\bigl(S_{3,1}\mathbf{x}\bigr) = S_{3,-1}\left(S_{3,1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

$$= S_{3,-1} \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}.$$

(d)

$$S_{3,1}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence

$$S_{3,1}^3 = S_{3,1}S_{3,1}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) Again, using the formulas in Section 2.1 we have

$$S_{3,1}^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S_{3,1} \left(S_{3,1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = S_{3,1} \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix},$$

and hence

$$S_{3,1}^3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S_{3,1} \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix} . = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

(f) We have $S_{3,1}^m = 0$ for $m = 3$ and therefore for all $m \geq 3$. Hence

$$e^{S_{3,1}t} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bigl(I + (S_{3,1}t) + (S_{3,2}t)^2/2 + 0 + 0 + \cdots \bigr) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} + (t^2/2) \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + tx_2 + (t^2/2)x_3 \\ x_2 + tx_3 \\ x_3 \end{bmatrix}$$

We do not give the matrix calculation, but since

$$\begin{bmatrix} x_1 + tx_2 + (t^2/2)x_3 \\ x_2 + tx_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

the equivalent matrix computation produces the 3×3 matrix on the right-hand-side above.

(g) We have

$$(I + S_{3,1})^{-1} = I - S_{3,1} + S_{3,1}^2 - S_{3,1}^3 + \cdots$$

and $S_{3,1}^3 = 0$, we have

$$(I + S_{3,1})^{-1} = I - S_{3,1} + S_{3,1}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} .$$

To check that this is really the inverse we multiply

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \bigl(I + S_{3,1} \bigr) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which, indeed, is the identity matrix.

(2) (a) Assuming that $x \in \mathbb{R}$ is nonzero, simplify the expression

$$\bigl(x + x^{-1}\bigr)^3.$$

(b) Fix any value of $n \in \mathbb{Z}$; simplify the expression

$$\bigl(C_{n,1} + C_{n,-1}\bigr)^3.$$

(c) Using the binomial theorem

$$(x+y)^n = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n,$$

write an expression for

$$\bigl(x + x^{-1}\bigr)^n.$$

(d) Using the binomial theorem, write an expression for

$$\bigl(C_{n,1} + C_{n,-1}\bigr)^n.$$

Solution:

(a)

$$\bigl(x + x^{-1}\bigr)^3 = x^3 + 3x^2x^{-1} + 3xx^{-2} + x^{-3} = x^3 + 3x + 3x^{-1} + x^{-3}$$

(b) Since $C_{n,-1}$ is the inverse of $C_{n,1}$, these matrices (and all their powers commute), and hence, by part (a),

$$\bigl(C_{n,1} + C_{n,-1}\bigr)^3 = C_{n,1}^3 + 3C_{n,1} + 3C_{n,1}^{-1} + C_{n,1}^{-3}$$

or, equivalently

$$C_{n,3} + 3C_{n,1} + 3C_{n,-1} + C_{n,-3}$$

(c)

$$\bigl(x + x^{-1}\bigr)^n = x^n + nx^{n-1}x^{-1} + \binom{n}{2}x^{n-2}x^{-2} + \cdots + nxx^{-(n-1)} + x^{-n}$$

$$= x^n + nx^{n-2} + \binom{n}{2}x^{n-4} + \cdots + nx^{-n+2} + x^{-n}.$$

(d) We use part (c) in the same way as part (b) used part (a) to get

$$\bigl(C_{n,1} + C_{n,-1}\bigr)^n = C_{n,1}^n + nC_{n,1}^{n-2} + \binom{n}{2}C_{n,1}^{n-4} + \cdots + nC_{n,1}^{-n+2} + C_{n,1}^{-n},$$

or equivalently

$$C_{n,n} + nC_{n,n-2} + \binom{n}{2}C_{n,n-4} + \cdots + nC_{n,-n+2} + C_{n,-n}.$$

(3) Let

$$A = N_{\mathrm{ring},2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, set $X_n = A^n$, i.e.,

$$X_0 = I_2, \quad X_1 = A, \quad X_2 = A^2, \quad \ldots$$

(a) Show that

$$X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

(b) Can you guess a formula for X_n based on these three examples? [There is no credit for this part, but you should take a new moments to see if there is some simple pattern to this sequence.]

(c) Show that for any $a, b \in \mathbb{R}$, we have

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$$

(d) Explain concisely why we may now conclude that for any $a \in \mathbb{R}$ we have

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 2a \\ 2a & 2a \end{bmatrix}$$

(e) Show that the next few terms in the sequence $\{X_n\}$ are

$$X_3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, X_4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}.$$

(f) Show that the $\{X_n\}$ satisfy the "two-term (matrix) recurrence equation"

$$X_{n+1} = X_n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and also the recurrence equation

(1) $$X_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_n, \quad n \geq 0$$

(g) Have we seen such a "two-term (matrix) recurrence equation" related to the Fibonacci numbers? What is this "two-term (matrix) recurrence equation"
related to the Fibonacci numbers?

(h) Are you surprised that, in view of the fact that

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, X_3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, X_4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}.$$

that

$$X_0 \neq \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}?$$

[No credit for your answer to this question, just say whether or not you were surprised.] Explain why, in retrospect, you shouldn't be very surprised. [Full credit for your explanation here.]

Solution:

(a) The values of X_0, X_1 are I_2, A which we know. In addition,

$$X_2 = A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

(b) All answers are correct.

(c) Each entry of the product is $(a, a) \cdot (b, b) = 2ab$.

(d) Set $b = 1$; in this case $2ab = 2a$.

(e) Since X_2 has all its entries equal to 2, X_3 has all its entries equal to 4, by part (d), and then X_4 has all its entries equal to 8.

(f) For $n \geq 0$ we have

$$X_{n+1} = A^{n+1} = A^n A = X_n A,$$

and similarly

$$X_{n+1} = A^{n+1} = AA^n = AX_n.$$

(g) The two-term Finbonacci reccurence is

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

(h) I find this a little surprising. However, since A is not invertible, we can't expect that

$$X_0 = X_1 A^{-1}$$

really makes much sense, i.e., we cannot expect that

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

makes sense and should have all entries being the same.

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