CPSC 303 HOMEWORK 6 SOLUTIONS, SPRING 2020

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(1) Let $p(x)$ be any nonzero polynomial such that $p(2) = 0$, i.e., such that $p(x) = (x-2)q(x)$ for some polynomial $q(x)$.
   (a) Differentiate the equation $p(x) = (x-2)q(x)$ to get a formula for $p'(x)$ in terms of $q(x)$ and $q'(x)$.
   (b) Argue that if in addition $p'(2) = 0$, then $q(2) = 0$.
   (c) Argue that if $p(2) = 0$ and $p'(2) = 0$ (as in the previous part), then $p(x) = (x-2)^2 r(x)$ for some polynomial $r(x)$.
   (d) Argue that if in addition, $p''(2) = 0$, then $p(x) = (x-2)^3 s(x)$ for some polynomial $s(x)$.
   (e) Argue that for any twice differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, there is a unique polynomial of the form $p(x) = c_0 + c_1 x + c_2 x^2$ such that $p(2) = f(2)$, $p'(2) = f'(2)$, and $p''(2) = f''(2)$.

Solution:

(a) $p'(x) = q(x) + (x-2)q'(x)$.
(b) $p'(2) = q(2) + 0 \cdot q'(2) = q(2)$; hence $p'(2) = 0$ implies that $q(2) = 0$.
(c) Since $q(2) = 0$, we have $q(x) = (x-2)r(x)$ for some polynomial $r(x)$, and hence $p(x) = (x-2)q(x) = (x-2)^2 r(x)$.
(d) Since $p(x) = (x-2)^2 r(x)$, we have

$$p'(x) = 2(x-2)r(x) + (x-2)^2 r'(x),$$

and

$$p''(x) = 2r(x) + \text{ other stuff divisible by } x-2.$$

Hence $p''(2) = 2r(2)$, and so $p''(2) = 0$ implies that $r(x)$ is divisible by $x-2$, and hence $r(x) = (x-2)s(x)$ for some polynomial $s(x)$. Hence $p(x) = (x-2)^3 s(x)$.
(e) The equation $p(2) = f(2)$ gives a linear equation

$$c_0 + 2c_1 + 4c_2 = f(2)$$

for c_0, c_1, c_2; similarly the equations $p'(2) = f'(2)$ and $p''(2) = f''(2)$ give two more linear equations for c_0, c_1, c_2. In total, this gives us a 3×3 linear system, so it suffices to show that the homogeneous system has a unique solution; the homogeneous system is the set of equations

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in c_0, c_1, c_2 such that $p(x) = c_0 + c_1 x + c_2 x^2$ satisfies $p(2) = p'(2) = p''(2) = 0$. But if $p(x)$ is nonzero, then $p(x)$ must be divisible by $(x-2)^3$, which is impossible since p is of degree at most two.

There are other ways of solving this problem. One is to notice that the 3×3 linear system is of the form

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(2) \\ f'(2) \\ f''(2) \end{bmatrix},$$

Which has a unique solution since the above 3×3 matrix is upper triangular with nonzero diagonal entries.

(2) (a) Problem 15 of Chapter 10 of [A&G], part (a); this has to be done by hand using the "Error in Polynomial Interpolation" formula, a.k.a. the "Generalized Remainder Term" formula.
(b) Problem 15 of Chapter 10 of [A&G], part (b).
(c) Find the largest value and the average value of the error $|e^x - p_2(x)|$ for $x = i/1000$ for $i = 0, 1, \ldots, 1000$.
(d) What is the ratio of the largest value of $|e^x - p_2(x)|$ in part (c) to the upper upper bound of part (a)? What about the same, where the largest value is replaced with the average value?

Solution:

(a) By the "error in polynomial interpolation" formula, the error is bounded by

$$\frac{M}{3!} \max_{0 \le x \le 1} \bigl|(x - x_0)(x - x_1)(x - x_2)\bigr| \tag{1}$$

where M is a bound on the third derivative of e^x for $0 \le x \le 1$. First, since $(e^x)''' = e^x$ which at most e on $0 \le x \le 1$, we can take $M = e$ (and that is the best that we can do). Next let

$$u(x) = (x - x_0)(x - x_1)(x - x_2) = x(x - 1/2)(x - 1) = x^3 - (3/2)x^2 + (1/2)x;$$

to find its maximum over $0 \le x \le 1$ we evaluate u at the endpoints (namely $x = 0, 1$, where $u(x) = 0$), and we solve $u'(x) = 0$, calculating that

$$u'(x) = 3x^2 - 3x + 1/2,$$

whose roots are $(3 \pm \sqrt{3})/2$; at these values of x we have $u(x)$ is $\pm\sqrt{3}/36$ (calculating by hand) or $0.04811\ldots$ (by calculator/computer). Hence we have

$$\max_{0 \le x \le 1} \bigl|(x - x_0)(x - x_1)(x - x_2)\bigr| = \sqrt{3}/36 = 0.04811\ldots,$$

and hence (1) becomes

$$\frac{e}{3!}\sqrt{3}/36 = 0.021797\ldots$$

(b) There are, of course, a number of ways of doing this. Let's use Lagrange's method:

$$p_2(x) = \frac{(x - 1/2)(x - 1)}{(0 - 1/2)(0 - 1}e^0 + \frac{(x - 0)(x - 1)}{(1/2 - 0)(1 - 1/2}e^{1/2} + \frac{(x - 0)(x - 1/2)}{(1 - 0)(1 - 1/2}e^1$$
$$= (2x^2 - 3x + 1)e^0 + (-4x^2 + 4x)e^{1/2} + (2x^2 - x)e^1$$
$$= 0.841678572\, x^2 + 0.876603256\, x + 1.$$

(c) We run:

```
clear
x = 0 : 0.001 : 1;
y = 0.841678572 * x.^2 + 0.876603256 * x + 1 - exp(x);
max(abs(y)), mean(abs(y))
```

and MATLAB returns 0.0144 and 0.0087.

(d) So the maximum computed error compared with the theoretical bound is roughly $0.0144/0.0218 \approx 66\%$, and the same for the average value is roughly $0.0087/0.0218 \approx 40\%$.

(3) (a) Find a linear function $g(x)$ taking $[-1,1]$ to $[0,1]$, i.e., such that $g(-1)=0$ and $g(1)=1$.

(b) Let x_0, x_1, x_2 be the three Chebyshev points of $[-1,1]$ (i.e., [A&G], first displayed formula in Section 10.6, with $n=2$). Find an exact formula for x_0, x_1, x_2 using the fact that

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

(c) What are the values of $y_i = g(x_i)$?

(d) How do the y_0, y_1, y_2 relate to the second and third displayed formulas in Section 10.6 (i.e., the ones involving a and b)?

(e) What is the maximum value of the function

$$h(y) = \big|(y-y_0)(y-y_1)(y-y_2)\big|$$

over the interval $y \in [0,1]$? Check this by a MATLAB calculation over all $y = i/1000$ for $i = 0, 1, \ldots, 1000$.

(f) What is the maximum value of the function

$$h(y) = \big|(y-y_0)(y-y_1)(y-y_2)\big|$$

over the interval $y \in [0,1]$? Justify your answer with a calculation by hand, i.e., that uses neither computer nor a calculator.

Solution:

(a) Solving for $g(x) = c_0 + c_1 x$ we find $g(x) = x/2 + 1/2$.

(b) The formula for $\cos(3\theta)$ implies that

$$T_3(x) = 4x^3 - 3x.$$

Solving $4x^3 - 3x = 0$ gives $x = 0$ or $4x^2 = 3$, i.e., $x = \pm\sqrt{3}/2$. [Do you remember $\pm\sqrt{3}/2$ from high school trig?] Hence we have x_0, x_1, x_2 are $-\sqrt{3}/2, 0, \sqrt{3}/2$.

(c) We have

$$y_0 = -\sqrt{3}/4 + 1/2,\ y_1 = 1/2,\ y_2 = \sqrt{3}/4 + 1/2$$

(d) This part of Section 10.6 describes how to map the usual Chebyshev points, on $[-1,1]$, to the Chebyshev points on $[a,b]$; hence the case in this homework problem corresponds to the formula where $a = 0$ and $b = 1$.

(e) We use the MATLAB code

```
clear
y = 0 : 0.001 : 1
h = abs(  (y + sqrt(3)/4 - 1/2) .* (y-1/2) .* ( y - sqrt(3)/4 - 1/2) )
max(h)
```

which returns 0.0313.

(f) We have $T_3(x)$ takes all values between -1 and 1 for x ranging over $[-1,1]$. Since the inverse function of g is $g^{-1}(y)=2y-1$ (mapping $[0,1]$ to $[-1,1]$), it follows that $p(y)=T_3(g^{-1}(y))$ takes on all values between -1 and 1 for y ranging over $[0,1]$. Since

$$p(y)=4(2y-1)^3-3(2y-1)=32y^3+\text{ lower order terms,}$$

we have that

$$p(y)=32(y-y_0)(y-y_1)(y-y_2)$$

and this polynomial takes on all values between -1 and 1 for y ranging over $[0,1]$. Hence

$$v(y)=p(y)/32=(y-y_0)(y-y_1)(y-y_2)$$

takes on all values between $-1/32$ and $1/32$ for y ranging over $[0,1]$. Hence $h(y)=|v(y)|$ has maximum value $1/32\approx 0.0313$ for $y\in[0,1]$.

(4) Do Problem 2 again, but instead of $p_2(x)$ that interpolates e^x at $0, 1/2, 1$, replace the points $0, 1/2, 1$ with the three Chebyshev points for $[0, 1]$ that you found in Problem 3. ALSO (for part (e) of this question): how does the bound in part (a) and the largest and average computed errors in part (c) of this problem—using Chebyshev points—compare with those in Problem 2?

Solution:

(a) We get the similar error bound as in Problem 2:

$$\frac{M}{3!} \max_{0 \le x \le 1} \big|(x - x_0)(x - x_1)(x - x_2)\big|$$

with $M = e$, but the maximum of $\big|(x - x_0)(x - x_1)(x - x_2)\big|$ is $1/32$, giving a bound of $e(1/6)(1/32) \approx 0.0141577$.

(b) This time let's just use MATLAB, similarly to Problem 3 on Homework 5 and the handout "Remarks on Divided Differences" (see the code there, pages 14 and 15):

```
clear
values(1) = -sqrt(3)/4 + 1/2
values(2) = 1/2
values(3) = sqrt(3)/4 + 1/2

e_to_values = exp(values)
p = polyfit( values , e_to_values , 2)     % p has coefs c0,..,c2
```

This gives `p` as `0.8373 0.8634 1.0077`, i.e., $p_2(x)$ is the polynomial $c_0 + xc_1 + x^2c_2$ with $c_0 = 0.8373$, $c_1 = 0.8634$, and $c_2 = 1.0077$. [Notice that this is still pretty close to the polynomial found in Problem 2.]

(c) We continue with

```
x = 0 : 0.001 : 1 ;

err_in_interp =   abs( polyval(p,x) - exp(x) );

max(err_in_interp), mean(err_in_interp)
```

and MATLAB reports 0.0099 and 0.0054 for the max and average error in interpolation.

(d) We continue with

```
% these are the max and mean errors relative to the above bound 0.0141577

max(err_in_interp)/0.0141577, mean(err_in_interp)/0.0141577
```

which MATLAB reports—for the max and average error in interpolation relative to the upper bound above of 0.0141577—as roughly 70% and 38%.

(e) The upper bounds are 0.021797 versus 0.0141577 which is a ratio of roughly 1.540. The max computed errors in interpolation are are 0.0144 versus 0.0099 which is a ratio of roughly 1.45. The average computed errors in interpolation are are 0.0087 versus 0.0054 which is a ratio of roughly 1.61.

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