CPSC 303 HOMEWORK 5 SOLUTIONS, SPRING 2020

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(1) (a) For which i does MATLAB report 1.0000e-i for 10^{-i}?
  (b) For which i does MATLAB report 0 for 10^{-i}?
  (c) Why does MATLAB report 10^{-323} as 9.8813e-324 as opposed to 1.0000e-323 ? [Hint: Is 10^{-323} a normal number or a subnormal number in double precision?]
  (d) One can check (but you don't have to) that MATLAB report

```
10^(-217) * 10^20 * 10^197 - 1
```

as 0, and if $217, 197$ above are replaced with $n, n-20$ with $n = 1, \ldots, 300$ then MATLAB reports this values as within $\pm 2.2204 \times 10^{-16}$; this seems reasonable, since this reflects roughly 53-bits of precision. But why does MATLAB report

```
10^(-317) * 10^20 * 10^297 - 1
```

as 2.3069e-07? [Hint: Are any of the numbers in this line of code subnormal?]

Solution:

(a) $i = 300, \ldots, 318$, assuming that your version of MATLAB is running (what I believe to be) compliant double precision, e.g., on a (modern) laptop. I understand that some online version(s) of MATLAB will give the answer $i = 300, \ldots, 323$.

This is a terrific news. This means that either (1) some versions of MATLAB in current use are not double precision compliant, or (2) double precision standards don't require the same expression to evaluate to the same thing.

(b) $i \geq 324$.

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(c) 10^{-323} is $\geq 2^{-1074}$ and $< 2^{-1022}$ (which are, respectively, roughly 4.49×10^{-324} and 2.2×10^{-308}) and is therefore subnormal [and close to the lower limit]. Hence double precision only gives a few bits of precision in describing 10^{-323}. [Note: since MATLAB is reporting numbers to 5 digits of precision, it is reasonable that numbers the 10^{-i} for $i \geq -318$ be reported accurately, since thees are roughly 5 powers of 10 away from the lower limit of 2^{-1074}.]

(d) 10^{-317} is subnormal, and hence your error is larger than for normal numbers. [More precisely: 10^{-317} is within 10^7 of the lower limit of subnormal numbers; hence the error of magnitude roughly 10^{-7} occurs because of imprecision after roughly 7 digits.]

(2) Problem 2 (in the Exercise section) of the handout "Remarks on Divided Differences."

Solution:

(a) Since cos is always between -1 and 1, we have

$$|R_7(x)| = |x|^7 |\cos\xi|/7! \leq |x|^7/7! \leq 1/7!$$

since $|x| \leq 1$.

(b) MATLAB reports the largest error as 1.9568e-04 [which is very close to .00019841 . . .].

(3) Problem 3 (in the Exercise section) of the handout "Remarks on Divided Differences."

Solution:

(a) This gives the 6 Chebyshev values for $x_0, \ldots, x_5$ of

```
cheb =

    0.9659    0.7071    0.2588   -0.2588   -0.7071   -0.9659
```

(b) MATLAB reports this as 0.0313.

(c) By the error in polynomial interpolation formula, the error (or "remainder") term equals

$$(x - x_0) \ldots (x - x_5) \frac{f^{(6)}(\xi)}{6!}$$

where ξ is some real value in any interval containing x and $x_0, \ldots, x_5$. Since $f^{(6)}(x) = -sinx$, we have $|f^{(6)}(\xi)| \leq 1$, and hence the absolute value of the error is at most

$$\left| \frac{(x - x_0)(x - x_1) \ldots (x - x_5)}{6!} \right|$$

(d) The largest value of the numerator above is 0.0313, and so the error is bounded by 0.0313/6!, which is roughly 4.34e-05.

(e) The largest error reported is 5.9884e-06. [Note: of course, this is better than the uppoer bound of 4.34e-05 from the previous part; this is due to the fact that $|f^{(6)}(\xi)|$ is at most 1, but apparently much smaller, at least when $|(x - x_0)(x - x_1) \ldots (x - x_5)|$ attains its largest values (of rougly 0.0313).

(4) Problem 4 (in the Exercise section) of the handout "Remarks on Divided Differences."

Solution:

(a) Since $v(x) = x^6$, the largest value of $v(x)$ over $x \in [-1, 1]$ is 1.
(b) Since $v(x) = (x^3 - x)^2$, we have

$$v'(x) = 2(x^3 - x)(3x^2 - 1) = 2x(x^2 - 1)3(x^2 - 1/3).$$

It follows that $v'(x) = 0$ iff $x = 0, \pm 1, \pm 1/\sqrt{3}$. We have

$$v(0) = v(\pm 1) = 0, \quad v(\pm 1/\sqrt{3}) = (1 - 1/3)^2(1/3) = 4/27.$$

The only other possible maximum of $v(x)$ is one attained at an end-point of $[-1, 1]$, but at these points v is $v(\pm 1) = 0$. Hence over $[-1, 1]$, the maximum value of $v(x)$ is attained at $\pm 1/\sqrt{3}$, where this maximum value is

$$4/27 = 0.148148\ldots$$

(c) (From the code in part (c) of Problem 3:) 0.0313.
(d) MATLAB reports `max(abs(v)) - 1/32` (using the code in part (c) of Problem 3) as 8.3267e-17.
(e) Part (b) improves over part (a) by a factor of $4/27 = 0.148148\ldots$. Part (c) improves over part (b) by a factor of

$$(1/32)/(4/27) = 27/128 = 0.21\ldots$$

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