CPSC 303 HOMEWORK 3 SOLUTIONS, SPRING 2020

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(1) Use the formulas in Section 6 to find the 1-norm, 2-norm, and ∞-norm of the matrix
$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}.$$

Solution:

The sum of the absolute values of each row is 7; hence $\|A\|_\infty = 7$.
The sum of the absolute values of each column is 7; hence $\|A\|_1 = 7$.
We have
$$A^T A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 24 \\ 24 & 25 \end{bmatrix}$$
The eigenvalues of $A^T A$ are given by solving
$$0 = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 25 & 24 \\ 24 & 25 \end{bmatrix} \right) = (\lambda - 25)^2 - 24^2$$
which yields $\lambda = 25 \pm 24 = 49, 1$. Hence
$$\|A\|_2 = \sqrt{49} = 7.$$

Note that a similar computation shows that for any $\alpha, \beta \in \mathbb{R}$, the matrix
$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$
has eigenvector $(1, 1)$ with eigenvalue $\alpha + \beta$, and eigenvector $(1, -1)$ with eigenvalue $\alpha - \beta$. This makes finding the eigenpairs of real, symmetric 2×2 matrices easy. There is no such simple formula for such $n \times n$ matrices with $n \geq 3$.

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(2) Use the formula in Exercise 2 to find the 1-norm, 2-norm, and ∞-condition number of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}.$$

Solution:

We have

$$A^{-1} = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{9-16}\begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3/7 & 4/7 \\ 4/7 & -3/7 \end{bmatrix}$$

Hence the formula in Exercise 2 shows that for $p = 1, 2, \infty$ we have

$$\|A\|_p = |3| + |4| = 7, \quad \|A^{-1}\|_p = |-3/7| + |4/7| = 1,$$

and therefore

$$\mathrm{cond}_p(A) = \|A\|_p \left\|A^{-1}\right\|_p = 7 \cdot 1 = 7$$

for $p = 1, 2, \infty$.

(3) Consider the system in Example 2.2: $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 10^7 & 0 \\ 0 & 3 \end{bmatrix}.$$

Show your work in the following calculations:

(a) If $\mathbf{b} = [1\ 0]^T$, what is $\mathbf{x} = A^{-1}\mathbf{b}$?

(b) Let $y \in \mathbb{R}$ be any number. If $\widehat{\mathbf{b}} = [1\ y]^T$, what is $\widehat{\mathbf{x}} = A^{-1}\widehat{\mathbf{b}}$?

(c) Show that the relative error between $\widehat{\mathbf{b}}$ and $\mathbf{b}$ (see Definition 3.1) is

$$\mathrm{Rel}_\infty(\widehat{\mathbf{b}}, \mathbf{b}) = |y|.$$

(d) Show that

$$\mathrm{Rel}_\infty(\widehat{\mathbf{x}}, \mathbf{x}) = |y| 10^7/3.$$

(e) How does the ratio of the relative errors in the previous two parts relate to the ∞-condition number of A?

(f) Show that

$$\mathrm{Rel}_\infty(\mathbf{b}, \widehat{\mathbf{b}}) = \frac{|y|}{\max(1, |y|)}$$

(g) Show that

$$\mathrm{Rel}_\infty(\mathbf{x}, \widehat{\mathbf{x}}) = \frac{|y|/3}{\max(10^{-7}, |y|/3)}$$

(h) Compute the ratio of the relative errors in the previous two parts for the four values $y = 10, 1, 10^{-1}, 10^{-8}$; how do these values relate to the ∞-condition number of A?

Solution:

(a)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 10^7 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10^{-7} & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10^{-7} \\ 0 \end{bmatrix}.$$

(b)

$$\widehat{\mathbf{x}} = A^{-1}\widehat{\mathbf{b}} = \begin{bmatrix} 10^{-7} & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} 10^{-7} \\ y/3 \end{bmatrix}.$$

(c)

$$\mathrm{Rel}_\infty(\widehat{\mathbf{b}}, \mathbf{b}) = \frac{\|\widehat{\mathbf{b}} - \mathbf{b}\|_\infty}{\|\mathbf{b}\|_\infty} = \frac{\|(0, y)\|_\infty}{\|(1, 0)\|_\infty} = \frac{|y|}{1} = |y|$$

(d)

$$\mathrm{Rel}_p(\widehat{\mathbf{x}}, \mathbf{x}) = \frac{\|\widehat{\mathbf{x}} - \mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \frac{\|(0, y/3)\|_\infty}{\|(10^{-7}, 0)\|_\infty} = |y|\, 10^7/3$$

(e) Since A, A^{-1} are diagonal, their condition numbers (with respect to the ∞-norm, but also any p norm) is the largest absolute value of their entries, and hence

$$\mathrm{cond}_\infty(A) = \|A\|_\infty \, \|A^{-1}\|_\infty = 10^7 \cdot (1/3) = 10^7/3.$$

It follows that the ratio of the relative errors above is precisely the condition number (i.e., we have found a worst case relative error "in the solution of $A\mathbf{x} = \mathbf{b}$").

(f)

$$\mathrm{Rel}_\infty(\mathbf{b}, \widehat{\mathbf{b}}) = \frac{\|\widehat{\mathbf{b}} - \mathbf{b}\|_\infty}{\|\widehat{\mathbf{b}}\|_\infty} = \frac{\|(0,y)\|_\infty}{\|(1,y)\|_\infty} = \frac{|y|}{\max(1,|y|)}$$

(g)

$$\mathrm{Rel}_p(\mathbf{x}, \widehat{\mathbf{x}}) = \frac{\|\widehat{\mathbf{x}} - \mathbf{x}\|_\infty}{\|\widehat{\mathbf{x}}\|_\infty} = \frac{\|(0,y/3)\|_\infty}{\|(10^{-7},y/3)\|_\infty} = \frac{|y/3|}{\max(10^{-7},|y/3|)}$$

(h) The relative error is therefore

$$\frac{|y/3|}{\max(10^{-7},|y/3|)} \Bigg/ \frac{|y|}{\max(1,|y|)} = \frac{\max(1,|y|)}{3\max(10^{-7},|y/3|)}$$

It follows that

(i) For $y = 10$, or really any $|y| \geq 1$, the relative error is $|y|/(3|y/3|) = 1$.

(ii) For $y = 1, 10^{-1}$, or really any y with $1 \geq |y| \geq 3 \cdot 10^{-7}$, the relative error is $1/(3|y/3|) = 1/|y|$.

(iii) For $y = 10^{-8}$, or really any y with $|y| \leq 3 \cdot 10^{-7}$, the relative error is $1/(3 \cdot 10^{-7}) = 10^7/3$.

Hence it is only for $|y| \leq 3 \cdot 10^{-7}$ that the relative error is really the condition number $10^7/3$, i.e., the worst case. For $|y| \geq 1$ the condition number is 1, which is much smaller than the condition number; and for $|y|$ between $3 \cdot 10^{-7}$ and 1 the condition number decreases as $|y|$ increases, covering the full range of $10^7/3$ to 1.

(4) Let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, and let $M = \|\mathbf{v}\|_\infty$, i.e.,

$$M = \max\bigl(|v_1|, \ldots, |v_n|\bigr).$$

Let p be any real number with $p \neq 0$.

(a) **In at most 20 words,** explain why

$$M^p \leq |v_1|^p + \cdots + |v_n|^p.$$

(b) **In at most 20 words,** explain why

$$|v_1|^p + \cdots + |v_n|^p \leq M^p n.$$

(c) **In at most 20 words plus one or two formulas,** using parts (a) and (b), explain why

$$\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_p \leq n^{1/p}\|\mathbf{v}\|_\infty.$$

Solution:

(a) Each $|v_i|^p$ is non-negative, and one of them equals M^p.

(b) Each $|v_i|^p$ is at most M^p, and there are n values of i.

(c) Taking p-th roots in parts (a) and (b) gives

$$M \leq \bigl(|v_1|^p + \cdots + |v_n|^p\bigr)^{1/p} \leq n^{1/p} M;$$

hence [by the definition of $\|\mathbf{v}\|_p$ for $p = 1, \infty$]

$$\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_p \leq n^{1/p}\|\mathbf{v}\|_\infty.$$

(5) Let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that

$$v_1 \le v_2 \le \ldots \le v_n,$$

and let $\mathbf{1}$ denote the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$. Let $x \in \mathbb{R}$ be a variable. Let $v_{\mathrm{middle}} = (v_1 + v_n)/2$, and $r = (v_n - v_1)$.

(a) **In at most 20 words plus one or two formulas**, show that if $x < v_{\mathrm{middle}}$, then

$$|x - v_n| > r/2.$$

(b) **In at most 20 words plus one or two formulas**, show that if $x > v_{\mathrm{middle}}$, then

$$|x - v_1| > r/2.$$

(c) **In at most 20 words plus one or two formulas**, show that if $x = v_{\mathrm{middle}}$, then

$$|x - v_1| = |x - v_n| = r/2$$

(d) **In at most 20 words plus one or two formulas**, find the value of $x \in \mathbb{R}$ at which

$$f(x) = \|\mathbf{v} - x\mathbf{1}\|_\infty$$

attains its minimum.

Solution:

(a) We have

$$v_n - x = (v_n - v_{\mathrm{middle}}) + (v_{\mathrm{middle}} - x) > v_n - v_{\mathrm{middle}} = r/2,$$

and hence $|x - v_n| > r/2$ (since $r \ge 0$ since $v_n \ge v_1$).

(b) We have

$$x - v_1 = (x - v_{\mathrm{middle}}) + (v_{\mathrm{middle}} - v_1) > v_{\mathrm{middle}} - v_1 = r/2,$$

and hence $|x - v_1| > r/2$.

(c) We have

$$v_{\mathrm{middle}} - v_1 = v_n - v_{\mathrm{middle}} = r/2,$$

and hence

$$|v_{\mathrm{middle}} - v_1| = |v_{\mathrm{middle}} - v_n| = r/2.$$

(d) For any i, $v_1 \le v_i \le v_n$, and so for $x \in \mathbb{R}$,

$$\|\mathbf{v} - x\mathbf{1}\|_\infty = \max_i |v_i - x| = \max(|x - v_1|, |x - v_n|)$$

which greater than $r/2$ if $x \ne v_{\mathrm{middle}}$, by parts (a) and (b), and equals $r/2$ if $x = v_{\mathrm{middle}}$ by part (c).

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