CPSC 303 Homework 1 Solutions

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1. Consider the following experiment: set $x_0 = 1$, $x_1 = 1/2$, and define $x_2, x_3, \ldots$ by the formula

$$x_{n+2} = (3/2)x_{n+1} - (1/2)x_n.$$

Hence in exact arithmetic, $x_n = (1/2)^n$. Perform this experiment in MATLAB, to numerically compute $x_2, x_3, \ldots, x_{1200}$. At one point does this sequence cycle (repeat a pattern)? What is the pattern and what is its period?

[Hint: Look at Section 5.4 of the handout; you could cut and paste the code from there (you may need to do this line by line). If your numerical experiment is like mine, for x_n with n roughly 1075 you will see a repeating pattern

```
{[4.9407e-324]}    {[9.8813e-324]}    {[1.4822e-323]}    {[1.4822e-323]}
{[9.8813e-324]}    {[4.9407e-324]}    {[4.9407e-324]}    {[9.8813e-324]}
{[1.4822e-323]}    {[1.4822e-323]}    {[9.8813e-324]}    {[4.9407e-324]}
```

and so the period of this repeating pattern is 6.]

Solution:

MATLAB appears to computes the sequence x_n correctly for $n = 0, \ldots, 1074$, with x_n for $n = 1071$ to $n = 1074$ reported as

```
{[3.9525e-323]}    {[1.9763e-323]}    {[9.8813e-324]}    {[4.9407e-324]}
```

until the above pattern appears at $n = 1075, \ldots, 1086$; the period of the cycle is 6 [meaning that $x_{n+6} = x_n$ for $n \geq 1074$].

The following MATLAB code (adapted from Section 5.4 of the handout) shows this pattern:

```
clear
x{1} = 1;
x{2} = 1/2;
for i=3:1100, x{i} = (3/2)*x{i-1}-(1/2)*x{i-2}; end
x
```

[The vector x begins with `x{1}` since MATLAB doesn't allow the index 0 for (an array or) a cell array.]

The following is an additional step not required on the homework: To make sure that x_{n+1}/x_n is computed as 1/2 for $n \leq 1073$, you can check by running the following code after you have run the above code:

```
for i=2:1099, y{i}=x{i+1}/x{i}; end
y
```

which gives `0.5000` until $i = 1075$ (corresponding to x_{n+1}/x_n with $n = 1074$, at which point you get the sequence of period six:

```
{[1]}    {[2]}    {[1.5000]}    {[1]}    {[0.6667]}    {[0.5000]}
```

[It should not surprise you that the product of these six numbers is 1, since the pattern of x_n repeats every six iterations.]

2. Same question as (1), except with $x_0 = 1$, $x_1 = 1/3$, and

$$x_{n+2} = (4/3)x_{n+1} - (1/3)x_n$$

for $n = 0, 1, 2, \ldots$

Solution:

The general solution of

$$x_{n+2} = (4/3)x_{n+1} - (1/3)x_n$$

is

$$C_1 r_1^n + C_2 r_2^n$$

where r_1, r_2 are the two solutions to

$$r^2 = (4/3)r - (1/3),$$

i.e., $r = 1, 1/3$. Hence the exact solution with $x_0 = 1$ and $x_1 = 1/3$ is $x_n = (1/3)^n$.

The numerical experiment is very different:

```
clear
x{1} = 1;
x{2} = 1/3;
for i=3:100, x{i} = (4/3)*x{i-1}-(1/3)*x{i-2}; end
x
```

Imprecision gets very clear around `x{i}` with $i = 35, \ldots, 38$

```
{[5.8224e-17]}    {[1.8249e-17]}    {[4.9239e-18]}    {[4.8228e-19]}
```

(note the factor of 10 between the last two terms). Then `x{i}` remains fixed (to within the five digits reported) at -1.7385e-18 around $i \geq 48$.

The following is an additional step not required on the homework: What you are seeing is the finite precision error in dividing by 3 in the base-2 double precision computation. Roughly speaking, this looks like the solution

$$C_1 + C_2(1/3)^n$$

with $C_2 = 1$ and C_1 roughly -1.7385e-18; the truth is a bit more complicated, because the finite precision errors are more complicated and compound at each step. However, this model is still pretty close, as the following experiments show: to be safe, we compute x_n numerically until $n = 200$ (i.e., to $x\{201\}$); we set $C_1 = 1$ and $C_2 = x\{201\}$, so that $C_1 + C_2(1/3)^n$ should match $x_n = x\{n+1\}$ for $n = 0$ and $n = 200$. We now compute $y_n = C_1 + C_2(1/3)^n$, and see how close y_n and x_n are, by setting $r_n = y_n/x_n$ (r for "ratio") and computing $d_n = 1 - r_n$ (d for "difference"):

```
clear
x{1} = 1;
x{2} = 1/3;
for i=3:201, x{i} = (4/3)*x{i-1}-(1/3)*x{i-2}; end
C1 = x{200}
for i=0:200, y{i+1} = C1 + (1/3)^i; end
for i=0:200, r{i+1}=y{i+1}/x{i+1}; end
for i=0:200, d{i+1} = 1 - r{i+1}; end
r
```

We see that the model is extremely close: the sequence $d\{i\}$ seems lagest at $d\{38\}$ reported as 7.2164e-15, and is reported as 0 for $d\{i\}$ with $i = 1, 2, 3$ and $i \geq 69$.

3. Same questions as (1) and (2), except with $x_0 = 1$, $x_1 = r$, and

$$x_{n+2} = (1 + r)x_{n+1} - (r)x_n$$

for $n = 0, 1, 2, \ldots$, for the value $r = 1/4$.

Solution:

The exact solution is obtained by solving $z^2 = (1 + r)z - r$ which has roots $1, r$; therefore the general solution is

$$x_n = C_1 + C_2 r^n$$

and the special case $x_0 = 1$ and $x_1 = r$ is therefore $x_n = r^n$.

In the case of $r = 1/4$ you get the exact answer (since double precision works in base 2) until x_n gets to the limit of indistinguishability from 0 (in the exponent of the scientific notation base 2):

```
clear
x{1} = 1;
x{2} = 1/4;
for i=3:600, x{i} = (5/4)*x{i-1}-(1/4)*x{i-2}; end
x
```

which has $x\{i\}$ equal 4.9407e-324 for $i = 538$, and 0 for $i \geq 539$.

4. Same question as (3) for $r = 1/5$.

Solution:

The formulas in Question 3 still hold, i.e.,

$$x_n = C_1 + C_2 r^n$$

for the general solution, and the special case $x_n = r^n$ when $x_0 = 1$ and $x_1 = r$ hold; this time $r = 1/5$. We then run the experiment:

```
clear
x{1} = 1;
x{2} = 1/5;
for i=3:40, x{i} = (6/5)*x{i-1}-(1/5)*x{i-2}; end
x
```

which (because of the division by 5, similar to Problem 2 with division by 3 in the base 2 double precision) eventually converges to -4.0416e-17, at roughly $x\{32\}$, for similar reasons as in Problem 2.

5. (a) In MATLAB, perform the numerical experiment $x_0 = 1$, $x_1 = 1/3$, and

$$x_{n+2} = (10/3)x_{n+1} - x_n$$

(for $n = 0, 1, 2, \ldots$). Describe the results (e.g., what happens for some interations the beginning, at the end, and somewhere where a transition occurs).

(b) What is the general solution to the recurrence $x_{n+2} = (10/3)x_{n+1} - x_n$?

(c) Explain why this general solution produces a numerical pattern like the one you see.

Solution:

(a) Examining the results of

```
clear
x{1} = 1;
x{2} = 1/3;
for i=3:800, x{i} = (10/3)*x{i-1}-x{i-2}; end
x
```

we see that the sequence begins to decay like $(1/3)^n$, then makes a sort of "transition"

```
  Columns 19 through 22

    {[5.9170e-09]}    {[1.0868e-08]}    {[3.0309e-08]}    {[9.0162e-08]}

  Columns 23 through 26

    {[2.7023e-07]}    {[8.1061e-07]}    {[2.4318e-06]}    {[7.2954e-06]}
```

whereupon it increases by factors of 3 until it hits the upper limit of double precision:

```
  Columns 681 through 685

    {[2.3849e+307]}    {[7.1547e+307]}    {[Inf]}    {[Inf]}    {[NaN]}
```

The transition that $x\{n+1\} = x_n$ makes from decreasing by 1/3 to increasing by 3 occurs around $n + 1 = 15, \ldots, 22$, as we can see by examining:

```
  Columns 15 through 18

    {[2.0912e-07]}    {[6.9815e-08]}    {[2.3601e-08]}    {[8.8555e-09]}

  Columns 19 through 22

    {[5.9170e-09]}    {[1.0868e-08]}    {[3.0309e-08]}    {[9.0162e-08]}
```

(b) The general solution to $x_{n+2} = (10/3)x_{n+1} - x_n$ is given by solving $r^2 = (10/3)r - 1$, whose solutions are $r = 3, 1/3$ and hence the general solution is

$$C_1 3^n + C_2(1/3)^n.$$

(c) The special case $x_0 = 1$, $x_1 = 1/3$, is therefore ($C_1 = 0$ and $C_2 = 1$ and) $r^n = (1/3)^n$.

(d) Because of the division by 3, the numerical experiment in double precision gives a solution that looks more like $C_2 = 1$ and C_1 very small but nonzero. Hence for small n the x_n is roughly $(1/3)^n$, where $C_2(1/3)^n$ is the dominant term, while for large n the dominant term $C_1 3^n$, which is what x_n looks like.

For large n, x_n takes on the value of Inf (intuitively meaning $+\infty$) twice, at this point the next value is $(10/3)(+\infty) - (+\infty)$, i.e., Inf minus Inf, which is NaN (not a number).

The following is an additional step not required on the homework: It is interesting to try to model the numerical computation of x_n by finding C_1 (with $C_2 = 1$, since this is what you observe at for small n) that match the computation.

Since $x\{682\}$ is 7.1547e+307, we can solve

$$x\{682\} = x_{681} = C_1 3^{681} = \texttt{7.1547e} + \texttt{307}$$

to solve for C_1. In MATLAB (and double precision), $3^{681} = \texttt{Inf}$, so in MATLAB we need to choose a value of n slightly smaller than 681 so that 3^{681} is still finite (in double precision). We have $3^{600} = \texttt{1.8739e} + \texttt{286}$, so we solve for C_1 in

$$C_1 3^{600} = x_{600} = x\{681\} = \texttt{2.3849e} + \texttt{307}.$$

Hence additional MATLAB code

```
C1 = x{601}/3^600
for i=0:700; y{i+1} = C1 * 3^i + (1/3)^i; end;
y
for i=0:700; z{i+1} = x{i+1}/y{i+1}; end;
z
```

sets `C1` to be 8.6103e-18, and then we see that the cell array `z` is 1 everywhere (!) until 3^n becomes `Inf`, and so `y` is a remarkably good approximation to `x` (or you could just compare x, y value-by-value.

Note that the value of `C1` is of a similar order of magnitude (i.e., -18 in this case) as the constants C_1 in the numerical experiments in the previous problems.

6. Same question as the previous question, for $x_0 = 1$, $x_1 = -1/3$, and the recurrence

$$x_{n+2} = (8/3)x_{n+1} + x_n.$$

In addition:

(a) Type the following calculations into MATLAB

```
(-4)^100001
(-4)^100000
(-4)^100001 + (-4)^100000
```

Explain (in 3-10 words) what the terms -Inf, Inf mean, and explain (in 5-15 words) why when you add -Inf and Inf you should get NaN (not a number).

(b) Why does the sequence in Problem (5) end in NaN repeating, while the one in Problem (6) ends in Inf (or possibly -Inf) repeating?

Solution:

There are two main differences with Question 5 and Question 6: first, the solution to $r^2 = (8/3)r + 1$ is $r = 3, -1/3$, so numerically we see something like $(-1/3)^n$ for small n and, as in Problem 5, $C_1 3^n$, for large n. And second, for very large n the numerical values of x_n remain `Inf`, since the recurrence is always adding two (positive) multiples of `Inf` (i.e., positive infinity), which remains `Inf`.

The code to run is

```
clear
x{1} = 1;
x{2} = -1/3;
for i=3:800, x{i} = (8/3)*x{i-1}+x{i-2}; end
x
```

The transition occurs around $i = 20$:

```
  Columns 14 through 17

    {[-6.2722e-07]}    {[2.0910e-07]}    {[-6.9628e-08]}    {[2.3421e-08]}

  Columns 18 through 21

    {[-7.1726e-09]}    {[4.2939e-09]}    {[4.2779e-09]}    {[1.5702e-08]}

  Columns 22 through 25

    {[4.6149e-08]}    {[1.3877e-07]}    {[4.1619e-07]}    {[1.2486e-06]}
```

The additional parts of Question 6 are meant to emphasize the difference `Inf`, `-Inf`, and `NaN` and arithmetic operations on these values.

(a) [The results are `-Inf`, `Inf`, and `NaN`.] `-Inf` means minus infinity; `Inf` means positive infinity; since ±`Inf` refer to anything beyond double precision, their sum can't be determined, and therefore called `NaN`.

(b) In Problem 6 we add positive infinity with a finite value, and then repeatedly add two positive infinities, producing the repeating values of `Inf`, whereas in Problem 5 we subtract two positive infinities, producing `NaN`, and then all future iterations become `NaN`'s as well.

The following is an additional step not required on the homework:

To see when numbers are beyond double precision, you can run the MATLAB code:

```
for i=1:350; [ 10^i, -10^i] , end
```

which shows that $\pm 10^{308}$ are the first powers of 10 designated as ±`Inf`. (Alternatively, you could look at the top of page 31, Section 2.4, of [A&G].)