HOMEWORK 7, CPSC 303, SPRING 2020

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The main goal of this homework is to develop some insight into the type linear algebra that we will need to discuss certain special differential equations, especially the heat equation.

For the rest of this course, all new material, aside from that in the textbook [A&G] will be given in the homeworks. These homeworks may also review some of the previous material.

At this point, given the fallout from COVID-19, I suggest that if you have limited bandwith or uncertain internet access, then you should download the textbook [A&G] and the course handouts now, especially the last handout on Energy and Splines, which we refer to here as [EnergySplines].

Please note: ALL OF THIS HOMEWORK MUST BE DONE BY HAND. You may not use MATLAB or any other computing device.

1. Some Fundamental Matrices of Interest to Us

In this section we introduce notation to describe two families of matrices of interest to us to understand certain differential equations. We have seen one of the families before when we discussed splines.

1.1. Matrices Describing Both Splines and Heat Transfer over a Rod.

For reasons that will become clear later, for any $n \in \mathbb{N} = \{1, 2, \ldots\}$ we define $N_{\mathrm{rod},n}$ to be the matrix $\mathbb{R}^{n\times n}$, i.e., the $n \times n$ square matrix

$$N_{\mathrm{rod},n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

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For example, we have

$$N_{\mathrm{rod},2}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad N_{\mathrm{rod},3}=\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix},\quad N_{\mathrm{rod},4}=\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{bmatrix}$$

One can alternatively describe $N_{\mathrm{rod},n}$ by its entries: for all $i,j\in[n]$, the (i,j)-th component of

$$\bigl(N_{\mathrm{rod},n}\bigr)_{i,j}=\begin{cases}1 & \text{if } i-j=\pm 1, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

We have already seen this matrix when we discussed splines; indeed, it is nothing more than the "off diagonal part" of the $n\times n$ matrix given at the top of page 343 (Section 11.3) of [A&G] in the special case $h_1=h_2=\cdots=h_n=1$. Furthermore, the full matrix in [A&G] at the top of page 343 (with $h_1=h_2=\cdots=h_n=1$) can be written more simply as

$$4I_n+N_{\mathrm{rod},n},$$

where I_n dentoes the $n\times n$ identity matrix. We use the term "rod" because we think of the interval $[A,B]$ ($[a,b]$ in the textbook) with arbitrary A and $B=A+n$ as a finite, one-dimensional rod.

There are some fundamental observations in [EnergySplines], Sections 4.2-4.4, and the corresponding [A&G] Section 11.3 regarding solving the equation

$$\Bigl(4I_n+N_{\mathrm{rod},n}\Bigr)\mathbf{c}=\boldsymbol{\psi},$$

where f is a function that we are modeling, like the profile of a car, and $\boldsymbol{\psi}$ was called in class the "energy of f," since $\boldsymbol{\psi}$ is essentially comprised of second divided differences of f (again, in the special case where $h_1=h_2=\cdots=h_n=1$). The observation is that

$$\Bigl(4I_n+N_{\mathrm{rod},n}\Bigr)^{-1}=(1/4)\bigl(I_n-(N/4)+(N/4)^2-(N/4)^3+\cdots\bigr)$$

which converges "geometrically" since $\|N/4\|_\infty\le 1/2$ (where we have written simply N for $N_{\mathrm{rod},n}$). As a consequence, the formula

$$\mathbf{c}=(1/4)\bigl(I_n-(N/4)+(N/4)^2-(N/4)^3+\cdots\bigr)\boldsymbol{\psi}$$

tells us that the effect of the values of $\boldsymbol{\psi}$ on $\mathbf{c}$ are "localized," in that for any $k\in\mathbb{Z}$, N^k is only nonzero on entries i,j with $|i-j|\le k$, and the term involving N^k, namely $(N/4)^k$, has ∞-norm at most $1/2^k$.

The matrix $N_{\mathrm{rod},n}$ can also be viewed as describing which integers in $[n]=\{1,\dots,n\}$ are "immediate neighbours" or "adjacent." This matrix is useful in *graph theory*, on the graph with vertices $1,\dots,n$ where we "joined by an edge" any two vertices differing by 1.

There are many interpretations and applications of this matrix.

Again, as in [EnergySplines] we use the (somewhat awkward choice of) letter N to suggest the notion that this matrix represents a relation between "neighbours" along a rod.

1.2. **Matrices Describing Heat Transfer over a (One-Dimensional) Ring.** Many computations turn out to be simpler when we "connect" the endpoints of the above rod, bending the rod to form a ring. This gives us the matrix

$$N_{\mathrm{ring},n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

which is just the matrix $N_{\mathrm{rod},n}$ with a 1 added to the top right and to the bottom left corners. This matrix as each column sum and each row sum equal to 2; it also has a *cyclic symmetry* that makes it a *Toeplitz matrix* (see the Wikipedia page on *Toeplitz Matrix*); we may return to Toeplitz matrices later.

For example, we have

$$N_{\mathrm{ring},2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad N_{\mathrm{ring},3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad N_{\mathrm{ring},4} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

2. Matrix Multiplication, Polynomials, Power Series, and Exponentiation Involving the Rod and Ring Matrices

In this section we will give the tools we need to carry out calculations with the rod and ring matrices defined above.

2.1. **Rod Matrices and Shift Operators.** Let us interpret the matrix $N_{\mathrm{rod},n}$ for a fixed n as an operator.

First, for any $n \in \mathbb{N}$, note that

$$N_{\mathrm{rod},n} = S_{n,1} + S_{n,-1},$$

where

$$S_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_{n,-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix};$$

hence $S_{n,1}$ is the nonzero part of $N_{\mathrm{rod},n}$ that lies above the diagonoal, and $S_{n,-1}$ the part below. However, $S_{n,1}$ has a simple interpretation as "shifting up by one," in the sense that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}$ (which, as always, we think of as a column vector),

$$S_{n,1}\,\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \\ 0 \end{bmatrix}$$

Hence the way that $S_{n,1}$ *operates* on a vector $\mathbf{x}$ is to move all its components up by one, and introduce a zero in the bottom component. Similarly we have

$$S_{n,-1}\,\mathbf{x} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1}, \end{bmatrix}$$

so $S_{n,-1}\mathbf{x}$ *operates* by shifting the components of $\mathbf{x}$ down by one and introduces a 0 on the top.

2.2. **Ring Matrices and Cyclic Shift Operators.** Working with ring matrices is much simpler, because they can be described as a sum of cyclic shift operators: indeed, for any $n \in \mathbb{N}$, note that

$$N_{\mathrm{ring},n} = C_{n,1} + C_{n,-1}, \tag{1}$$

where

$$C_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{n,-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly to the previous subsection, we have

$$C_{n,1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \\ x_1, \end{bmatrix} \tag{2}$$

and hence $C_{n,1}$ has the effect of "cyclically rotating the components of $\mathbf{x}$ up by one," taking the x_1 to be its bottom component, instead of the 0 that $S_{n,1}$ introduces. Similarly we have

$$C_{n,-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \tag{3}$$

2.3. **Ring Matrices and the $C_{n,\pm 1}$ are Easier to Work With than Rod Matrices and the $S_{n,\pm 1}$.** For many computations, it is easier to work with the $C_{n,\pm 1}$ than the $S_{n,\pm 1}$, and to see how powers and polynomials of

$$N_{\mathrm{ring},n} = C_{n,1} + C_{n,-1},$$

behave as opposed to

$$N_{\mathrm{rod},n} = S_{n,1} + S_{n,-1}.$$

For example, to interpret $N^2_{\mathrm{ring},n}$, we have

$$N^2_{\mathrm{ring},n} = \bigl(C_{n,1} + C_{n,-1}\bigr)^2$$

which equals

$$\bigl(C_{n,1} + C_{n,-1}\bigr)\bigl(C_{n,1} + C_{n,-1}\bigr) \tag{4}$$

To simiplify such an expression we note that

$$C^2_{n,1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_n \\ x_1 \\ x_2, \end{bmatrix}$$

which just cyclically shifts the components of $\mathbf{x}$ by 2. More generally, if for any $k \in \mathbb{N}$, if we set

$$C_{n,k} = C^k_{n,1},$$

then $C_{n,k}$ is the operator that cyclically rotates the components of a vector up by k; similarly for $C_{n,-k} = C_{n,-1}^k$. We similarly see that $C_{n,-1}C_{n,1}\mathbf{x}$ is just $\mathbf{x}$, and hence

$$C_{n,-1}C_{n,1} = C_{n,1}C_{n,-1} = I_n$$

the identity matrix. Hence all the $C_{n,\pm k}$ are invertible, and they all commute; setting $C_{n,0} = I_n$ (which makes sense, in that shifting by 0 does nothing to a vector), we conclude that

$$\bigl(C_{n,1} + C_{n,-1}\bigr)^2 = C_{n,2} + 2I_n + C_{n,-2}.$$

Furthermore, this can be seen as a manifestation of the identity

$$\bigl(x + x^{-1}\bigr)^2 = x^2 + 2 + x^{-2}.$$

Unfortunely, things are not as simple with the matrices or operators $S_{n,1}$ and $S_{n,-1}$. For one thing, $S_{n,1}$ is not invertible, since its bottom row consists entirely of 0's. And it is also not true that $S_{n,1}$ and $S_{n,-1}$ commute: i.e.,

$$S_{n,-1}S_{n,1} \neq S_{n,1}S_{n,-1}$$

(see the exercises). It turns out that for *positive* integers, k, one can set $S_{n,k} = S_{n,1}^k$, and this has as reasonable interpretation, just as does $S_{n,-k} = S_{n,-1}^k$, but when we multiply positive of powers of $S_{n,1}$ times those of $S_{n,-1}$, the operators we get will operate on a vector $\mathbf{x}$ like cyclic shifts except that some of the x_i are replaced by zero.

3. Review of Polynomials and Power Series in Matrices

We will use $\mathbb{R}^{n\times n}$ for a the set of $n \times n$ of matrices with real entries.

Recall that if A is an $n \times n$ matwe use the notation

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad A^3 = AAA,$$

and that all these matrices *commute*, i.e.,

$$A^sA^r = A^rA^s$$

for all $r, s \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

If $p(x) = c_0 + c_1x + \cdots + c_mx^m$ is a polynomial, we define

$$p(A) = c_0I + c_1A + \cdots + c_mA^m$$

where $I = I_n \in \mathbb{R}^{n\times n}$ is the $n \times n$ identity matrix (and where $c_i \in \mathbb{R}$ or $c_i \in \mathbb{C}$, i.e., the coefficients are real-valued or complex-valued).

If p, q are any two polynomials, then we easily see that $p(A)$ and $q(A)$ commute, i.e.,

$$p(A)q(A) = q(A)p(A),$$

and, moreover, if

$$r(x) = p(x)q(x)$$

is the usual product of polynomials, then we use the shorthand $r = pq$, and we have

$$r(A) = (pq)(A) = \bigl(p(A)\bigr)\bigl(q(A)\bigr).$$

Example 3.1. If $p(x) = x + 2$ and $q(x) = x + 1$, then

$$\begin{aligned}(A + 2I)(A + I) &= (A + I)(A + 2I) = A^2 + 3A + 2I \\ &= \text{“}(x^2 + 3x + 2)\,\big|_{x=A}\text{”} = \text{et cetera}\end{aligned}$$

In [EnergySplines], Section 4.2–4.4 we extended the notion of a polynomial of a matrix, A, to power series in A, such as

$$(I+A)^{-1} = I - A + A^2 - A^3 + \cdots \tag{5}$$

which is the matrix analog of

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

and where the power series in A converges whenever $\|A\|_\infty < 1$ (or similarly for other matrix norms), just as the power series in x converges whenever $|x| < 1$.

For reasons that will become clear when we work with differential equations, we will also be interested in the analog of the Taylor expansion for e^x about $x = 0$, namely

$$e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + \cdots$$

which is valid for all $x \in \mathbb{R}$, and its matrix analog

$$e^A = I + A + A^2/2 + A^3/3! + A^4/4! + \cdots$$

which converges for all $A \in \mathbb{R}^{n\times n}$. Roughly speaking, the reason is that the solution to the differential equation $dx/dt = ax$ for a constant $a \in \mathbb{R}$, and variables x, t is given by $x(t) = x(0)e^{at}$; similarly, the solution to the vector-valued differential equation $d\mathbf{x}/dt = A\mathbf{x}$ for an $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} = \mathbf{x}(t)$ taking values in $\mathbb{R}^n$ turns out to be

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0).$$

We will discuss this further in the next homework.

EXERCISES

(1) Let us make some calculations regarding the matrices $S_{n,1}$ and $S_{n,-1}$ in the case $n = 3$, where

$$S_{3,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_{3,-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Using regular matrix multiplication, compute the expressions $S_{3,1}S_{3,-1}$ and $S_{3,-1}S_{3,1}$. Are they equal?

(b) Compute the way $S_{3,1}S_{3,-1}$ operates on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ as

$$S_{3,1}\bigl(S_{3,-1}\mathbf{x}\bigr),$$

i.e., by first computing

$$S_{3,-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and then applying $S_{3,1}$ to the result.

(c) Similarly compute the way that $S_{3,-1}S_{3,1}$ operates on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

(d) Using matrix multiplications, compute the values of
$$S_{3,1}^2, \quad S_{3,1}^3.$$
(e) Compute the way that
$$S_{3,1}^2, \quad S_{3,1}^3$$
operate on a vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ by applying twice and three times, respectively, the operation of $S_{3,1}$ on $\mathbf{x}$.

(f) Compute the value of
$$e^{S_{3,1}t}$$
using the formula
$$e^{At} = I + (At) + (At)^2/2 + (At)^3/3! + \cdots$$
using either direct matrix calculations or operator calculations as above.

(g) Compute the value of $(I + S_{3,1})^{-1}$ using (5). Check your work by multiplying this result by $I + S_{3,1}$ to see that you get the identity matrix.

(2) (a) Assuming that $x \in \mathbb{R}$ is nonzero, simplify the expression
$$\left(x + x^{-1}\right)^3.$$
(b) Fix any value of $n \in \mathbb{Z}$; simplify the expression
$$\left(C_{n,1} + C_{n,-1}\right)^3.$$
(c) Using the binomial theorem
$$(x+y)^n = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n,$$
write an expression for
$$\left(x + x^{-1}\right)^n.$$
(d) Using the binomial theorem, write an expression for
$$\left(C_{n,1} + C_{n,-1}\right)^n.$$

(3) Let
$$A = N_{\mathrm{ring},2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$
and for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, set $X_n = A^n$, i.e.,
$$X_0 = I_2, \quad X_1 = A, \quad X_2 = A^2, \quad \ldots$$
(a) Show that
$$X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$
(b) Can you guess a formula for X_n based on these three examples? [There is no credit for this part, but you should take a new moments to see if there is some simple pattern to this sequence.]

(c) Show that for any $a, b \in \mathbb{R}$, we have

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$$

(d) Explain concisely why we may now conclude that for any $a \in \mathbb{R}$ we have

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 2a \\ 2a & 2a \end{bmatrix}$$

(e) Show that the next few terms in the sequence $\{X_n\}$ are

$$X_3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, X_4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}.$$

(f) Show that the $\{X_n\}$ satisfy the "two-term (matrix) recurrence equation"

$$X_{n+1} = X_n \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and also the recurrence equation

$$X_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_n, \quad n \geq 0 \tag{6}$$

(g) Have we seen such a "two-term (matrix) recurrence equation" related to the Fibonacci numbers? What is this "two-term (matrix) recurrence equation" related to the Fibonacci numbers?

(h) Are you surprised that, in view of the fact that

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, X_3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, X_4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}.$$

that

$$X_0 \neq \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}?$$

[No credit for your answer to this question, just say whether or not you were surprised.] Explain why, in retrospect, you shouldn't be very surprised. [Full credit for your explanation here.]

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