# HOMEWORK 6, CPSC 303, SPRING 2020

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may submit homework in groups of 1-3. You must acknowledge any sources (e.g., Wikipedia, other books, other articles) you have used beyond the textbook and the article(s) on the class website.

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In these exercises, "the handout" refers to the article "CPSC 303: Remarks on Divided Differences," on the CPSC 303 homepage.

Write your solutions to the following problems concisely. Marks will be deducted if you submit 80 values—or even 40 values—of any output the MATLAB code below produces.

Note that from this point in class and on the homework, we identify

$$(v_1, \ldots, v_n) \in \mathbb{R}^n$$

with the $n \times 1$ "column vector"

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

(1) Let $p(x)$ be any nonzero polynomial such that $p(2) = 0$, i.e., such that $p(x) = (x-2)q(x)$ for some polynomial $q(x)$.
  (a) Differentiate the equation $p(x) = (x-2)q(x)$ to get a formula for $p'(x)$ in terms of $q(x)$ and $q'(x)$.
  (b) Argue that if in addition $p'(2) = 0$, then $q(2) = 0$.
  (c) Argue that if $p(2) = 0$ and $p'(2) = 0$ (as in the previous part), then $p(x) = (x-2)^2 r(x)$ for some polynomial $r(x)$.

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(d) Argue that if in addition, $p''(2) = 0$, then $p(x) = (x-2)^3 s(x)$ for some polynomial $s(x)$.
(e) Argue that for any twice differentiable function $f : \mathbb{R} \to \mathbb{R}$, there is a unique polynomial of the form $p(x) = c_0 + c_1 x + c_2 x^2$ such that $p(2) = f(2)$, $p'(2) = f'(2)$, and $p''(2) = f''(2)$.

(2) (a) Problem 15 of Chapter 10 of [A&G], part (a); this has to be done by hand using the "Error in Polynomial Interpolation" formula, a.k.a. the "Generalized Remainder Term" formula.
(b) Problem 15 of Chapter 10 of [A&G], part (b).
(c) Find the largest value and the average value of the error $|e^x - p_2(x)|$ for $x = i/1000$ for $i = 0, 1, \ldots, 1000$.
(d) What is the ratio of the largest value of $|e^x - p_2(x)|$ in part (c) to the upper upper bound of part (a)? What about the same, where the largest value is replaced with the average value?

(3) (a) Find a linear function $g(x)$ taking $[-1, 1]$ to $[0, 1]$, i.e., such that $g(-1) = 0$ and $g(1) = 1$.
(b) Let $x_0, x_1, x_2$ be the three Chebyshev points of $[-1, 1]$ (i.e., [A&G], first displayed formula in Section 10.6, with $n = 2$). Find an exact formula for $x_0, x_1, x_2$ using the fact that

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

(c) What are the values of $y_i = g(x_i)$?
(d) How do the $y_0, y_1, y_2$ relate to the second and third displayed formulas in Section 10.6 (i.e., the ones involving $a$ and $b$)?
(e) What is the maximum value of the function

$$h(y) = \bigl|(y - y_0)(y - y_1)(y - y_2)\bigr|$$

over the interval $y \in [0, 1]$? Check this by a MATLAB calculation over all $y = i/1000$ for $i = 0, 1, \ldots, 1000$.
(f) What is the maximum value of the function

$$h(y) = \bigl|(y - y_0)(y - y_1)(y - y_2)\bigr|$$

over the interval $y \in [0, 1]$? Justify your answer with a calculation by hand, i.e., that uses neither computer nor a calculator.

(4) Do Problem 2 again, but instead of $p_2(x)$ that interpolates $e^x$ at $0, 1/2, 1$, replace the points $0, 1/2, 1$ with the three Chebyshev points for $[0, 1]$ that you found in Problem 3. ALSO (for part (e) of this question): how does the bound in part (a) and the largest and average computed errors in part (c) of this problem—using Chebyshev points—compare with those in Problem 2?

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