HOMEWORK 4, CPSC 303, SPRING 2020

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Please note:
(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may submit homework in groups of 1-3. You must acknowledge any sources (e.g., Wikipedia, other books, other articles) you have used beyond the textbook and the article(s) on the class website.

In these exercises, "the handout" refers to the article "CPSC 303: Normal and Subnormal Numbers in Double Precision," on the CPSC 303 homepage.

Write you solutions to the following problems concisely. Marks will be deducted if you submit 80 values—or even 40 values—of any output the MATLAB code below produces.

Note that from this point in class and on the homework, we identify

$$(v_1, \ldots, v_n) \in \mathbb{R}^n$$

with the $n \times 1$ "column vector"

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

(1) Problem 1 (in the Exercise section) of the handout "Normal and Subnormal Numbers in Double Precision."

(2) Problem 2 of the handout.

(3) Problem 3 of the handout.

(4) (a) Consider for a sequence $\ldots, y_{-1}, y_0, y_1, y_2, \ldots$, consider the recurrence relation:
$$y_{n+1} = y_n \quad \forall n \in \mathbb{Z}$$
(i.e., for all integers n). For for any $a \in \mathbb{R}$, what is the unique solution to this recurrence that satisfies the condition $y_0 = a$? Briefly justify your answer.

(b) Consider another recurrence relation:
$$y_{n+2} = 2y_{n+1} - y_n \quad \forall n \in \mathbb{Z}.$$
For any $a, b \in \mathbb{R}$, what is the unique solution to this recurrence that satisfies the condition $y_0 = a$ and $y_1 = a + b$? Briefly justify your answer.

(c) Consider another recurrence relation:

(1) $$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n \quad \forall n \in \mathbb{Z}.$$
Show that the following sequences satisfy this recurrence relation: $y_n = 1$, $\tilde{y}_n = n$, $\hat{y}_n = n^2$.

(d) Show that for any $C_1, C_2, C_3 \in \mathbb{R}$, the sequence
$$y_n = C_1 + C_2 n + C_3 n^2$$
satisfies (1) [you may use the previous part].

(e) If we define for each sequence
$$\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} = \{\ldots, y_{-1}, y_0, y_1, y_2, \ldots\}$$
a new sequence $D\mathbf{y}$ (known as the "(forward) difference of $\mathbf{y}$") defined by
$$(D\mathbf{y})_n = y_{n+1} - y_n,$$
we can "apply D twice" to get $D^2\mathbf{y}$, meaning $D(D\mathbf{y})$, given by
$$D(D\mathbf{y}) = (D\mathbf{y})_{n+1} - (D\mathbf{y})_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n).$$
Simplify this formula, and then compute a similarly simplified formula for $D^3\mathbf{y}$, meaning $D(D^2\mathbf{y})$. How do your formulas for relate to the previous parts (a)–(c) of this problem?

(f) For any $a, b \in \mathbb{R}$, what is the exact solution of (1) given the conditions $y_0 = a$, $y_1 = a + b$, $y_2 = a + 2b$?

(g) Run the following MATLAB code to test what happens numerically in the previous part for $a = \sqrt{2}$ and $b = \sqrt{7}$:

```
clear

n = 1
sequence_length = 30 * n

x{1}= sqrt(2)
x{2}= sqrt(2) + sqrt(7)
x{3}= sqrt(2) + 2 * sqrt(7)

for i=4:sequence_length, x{i}=3*x{i-1} - 3*x{i-2}+x{i-3}; end
```

```
for i=1:30; should_be_zero{i} = x{i*n}-sqrt(2)-(i*n-1)*sqrt(7); end
```

```
should_be_zero
```

and then run the entire code again (starting from the `clear` at the top) with $n = 10$, $n = 100$, $n = 1000$, and $n = 10000$. Explain why the cell array `should_be_zero` should be zero if MATLAB were computing in "exact" (or "infinite precision") arithmetic.

Roughly speaking, what do you think you are seeing numerically?

(h) Run the above code again for $n = 10000$ (i.e., 10,000), and at the bottom add the lines

```
for i=1:20; rough_effect_of_error{i} = should_be_zero{i} / (i*n-1)^2; end
rough_effect_of_error
```

Then run all of the above code with $n = 100000$ (i.e. 100,000), and $n = 1000000$ (i.e., 1,000,000). Is the description of the results as $C_1 + C_2 n + C_3 n^2$ with $C_1 = \sqrt{2}$, $C_2 = \sqrt{7}$, and C_3 a very small constant seem completely correct for $n = 10^4$? Is there any trend to how the apparent C_3 is behaving in the various entries of `rough_effect_of_error`? What about for $n = 10^5$? What about for $n = 10^6$? [Your answers may or may not be the same for these three values of n.]

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