Group Exercise 9 – SVD and Truncated SVD
Pre-Reading for Nov 3, 2017

Example 1 *The singular value decomposition of*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

is given by

$$A = U\Sigma V^T = \begin{pmatrix} 0.141 & 0.825 & -0.420 & -0.351 \\ 0.344 & 0.426 & 0.298 & 0.782 \\ 0.547 & 0.028 & 0.664 & -0.509 \\ 0.750 & -0.371 & -0.542 & 0.079 \end{pmatrix} \begin{pmatrix} 25.5 & 0 & 0 \\ 0 & 1.29 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.504 & 0.574 & 0.644 \\ -0.761 & -0.057 & 0.646 \\ 0.408 & -0.816 & 0.408 \end{pmatrix}.$$

Thus, we have $\sigma_1 = 25.5$, $\sigma_2 = 1.29$, and $\sigma_3 = 0$. A singular value of zero indicates that the matrix A is rank-deficient. In general, the rank of a matrix is equal to the number of nonzero singular values, which in this example is two.

The SVD of A can be written as

$$A = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix} V^T,$$

which leads to the economy-size SVD of A

$$A = U_1\Sigma_1 V^T = \begin{pmatrix} 0.141 & 0.825 & -0.420 \\ 0.344 & 0.426 & 0.298 \\ 0.547 & 0.028 & 0.664 \\ 0.750 & -0.371 & -0.542 \end{pmatrix} \begin{pmatrix} 25.5 & 0 & 0 \\ 0 & 1.29 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.504 & 0.574 & 0.644 \\ -0.761 & -0.057 & 0.646 \\ 0.408 & -0.816 & 0.408 \end{pmatrix}.$$

Given

$$\mathbf{b} = \begin{pmatrix} 13 \\ 14 \\ 15 \\ 16 \end{pmatrix}.$$

Using Gaussian elimination, we can transform the augmented matrix $(A|\mathbf{b})$ to the triangular form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & -3 & -6 & -38 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so the set of solutions to $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{pmatrix} \frac{3t-37}{3} \\ \frac{38-6t}{3} \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

The least squares solution of minimum Euclidean norm is given by

$$\mathbf{x} = \sum_{\sigma_i \neq 0} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i = \frac{\mathbf{u}_1^T \mathbf{b}}{\sigma_1} \mathbf{v}_1 + \frac{\mathbf{u}_2^T \mathbf{b}}{\sigma_2} \mathbf{v}_2$$

$$= \frac{\begin{pmatrix} 0.141 & 0.344 & 0.547 & 0.750 \end{pmatrix} \begin{pmatrix} 13 \\ 14 \\ 15 \\ 16 \end{pmatrix}}{25.5} \begin{pmatrix} 0.504 \\ 0.574 \\ 0.644 \end{pmatrix}$$

$$+ \frac{\begin{pmatrix} 0.825 & 0.426 & 0.028 & -0.371 \end{pmatrix} \begin{pmatrix} 13 \\ 14 \\ 15 \\ 16 \end{pmatrix}}{1.29} \begin{pmatrix} -0.761 \\ -0.057 \\ 0.646 \end{pmatrix}$$

$$= \begin{pmatrix} -6.0604 \\ 0.1108 \\ 6.2734 \end{pmatrix}$$

Example 2 *Consider*

$$A = \begin{pmatrix} 0.913 & 0.659 \\ 0.780 & 0.563 \\ 0.457 & 0.330 \end{pmatrix},$$

whose columns are nearly linearly dependent. The singular value decomposition of A is given by

$$A = U\Sigma V^T = \begin{pmatrix} 0.71058 & -0.26631 & -0.65127 \\ 0.60707 & -0.23592 & 0.75882 \\ 0.35573 & 0.93457 & 0.00597 \end{pmatrix} \begin{pmatrix} 1.58460 & 0 \\ 0 & 0.00011 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.811083 & 0.58528 \\ -0.58528 & 0.81083 \end{pmatrix}.$$

Using TSVD, we set the singular values below the cutoff tolerance of about 10^{-4} to 0. So, we set $\sigma_2 = 0$, thus, the effective rank of A is 1. We obtain the approximate matrix

$$A_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 1.58460 \begin{pmatrix} 0.71058 \\ 0.60707 \\ 0.35573 \end{pmatrix} \begin{pmatrix} 0.811083 & 0.58528 \end{pmatrix} = \begin{pmatrix} 0.91298 & 0.65902 \\ 0.77999 & 0.56302 \\ 0.45706 & 0.32992 \end{pmatrix},$$

which is an extremely close approximation to the original matrix A, since σ_2 is so tiny that the term associated with it makes almost no contribution to the sum. According to the Best Lower Rank Approximation Theorem, $\|A - A_1\|_2 = \sigma_2 = 0.00011$.

Example 3 *Consider* $A \in \mathbb{R}^{m\times n}$ *with* $m > n = \operatorname{rank}(A)$. *We can use the SVD of* A *to show that* $\operatorname{rank}(A) = \operatorname{rank}(A^T)$*:*

$A = U\Sigma V^T$ *with* $\Sigma = \left(\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline \mathbf{0} & \cdots & \mathbf{0} \end{array}\right)$. *The rank of* A *is the number of nonzero singular values, which is* n. *If* $A = U\Sigma V^T$ *is an SVD of* A, *then* $A^T = V\Sigma^T U^T$ *is an SVD of* A^T *with* $\Sigma^T = \left(\begin{array}{ccc|c} \sigma_1 & & & \mathbf{0}^T \\ & \ddots & & \vdots \\ & & \sigma_n & \mathbf{0}^T \end{array}\right)$. *Therefore,* A *and* A^T *have the same singular values. Hence* $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.