Group Exercise 10 – Convergence of Stationary Methods
Pre-Reading for Nov 10, 2017

Example 1 *The linear system*

$$
\begin{aligned}
10x_1 + x_2 + x_3 &= 12\,,\\
x_1 + 10x_2 + x_3 &= 12\,,\\
x_1 + x_2 + 10x_3 &= 12
\end{aligned}
\tag{1}
$$

has the unique solution $\mathbf{x}$ *with entries* $x_1 = x_2 = x_3 = 1$.

The coefficient matrix of the linear system (1) is

$$
A = \begin{pmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{pmatrix}.
$$

Since $|a_{ii}| = 10 > 2 = \sum_{j=1, j\neq i}^{3} |a_{ij}|$ *for* $i = 1, 2, 3$, *A is strictly diagonally dominant. Hence, Jacobi and Gauss–Seidel will converge for this matrix regardless of the starting vector* $\mathbf{x}^{(0)}$.

Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, *the Jacobi iteration reads*

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{12 - x_2^{(k)} - x_3^{(k)}}{10}\,,\\
x_2^{(k+1)} &= \frac{12 - x_1^{(k)} - x_3^{(k)}}{10}\,,\\
x_3^{(k+1)} &= \frac{12 - x_1^{(k)} - x_2^{(k)}}{10}\,,
\end{aligned}
$$

for $k = 0, 1, \ldots$. *We perform two iterations and obtain*

$$
\begin{aligned}
x_1^{(1)} &= \frac{6}{5} = 1.2\,, & x_1^{(2)} &= \frac{24}{25} = 0.96\\
x_2^{(1)} &= \frac{6}{5} = 1.2\,, & x_2^{(2)} &= \frac{24}{25} = 0.96\\
x_3^{(1)} &= \frac{6}{5} = 1.2\,, & x_3^{(2)} &= \frac{24}{25} = 0.96
\end{aligned}
$$

The l_1-error norm in the approximated solution $\mathbf{x}^{(2)}$ is

$$\left\|\mathbf{x}-\mathbf{x}^{(2)}\right\|_1 = \sum_{i=1}^{3}\left|x_i - x_i^{(2)}\right| = 3\,|1-0.96| = 0.12\,.$$

Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, the Gauss–Seidel iteration reads

$$\begin{aligned}
x_1^{(k+1)} &= \frac{12 - x_2^{(k)} - x_3^{(k)}}{10}\,,\\
x_2^{(k+1)} &= \frac{12 - x_1^{(k+1)} - x_3^{(k)}}{10}\,,\\
x_3^{(k+1)} &= \frac{12 - x_1^{(k+1)} - x_2^{(k+1)}}{10}\,,
\end{aligned}$$

for $k = 0, 1, \ldots$. We perform two iterations and obtain

$$\begin{aligned}
x_1^{(1)} &= \frac{6}{5} = 1.2\,, & x_1^{(2)} &= 0.9948\\
x_2^{(1)} &= \frac{27}{25} = 1.08\,, & x_2^{(2)} &= 1.00332\\
x_3^{(1)} &= \frac{243}{250} = 0.972\,, & x_3^{(2)} &= 1.000188
\end{aligned}$$

The l_1-error norm in the approximated solution $\mathbf{x}^{(2)}$ is

$$\left\|\mathbf{x}-\mathbf{x}^{(2)}\right\|_1 = \sum_{i=1}^{3}\left|x_i - x_i^{(2)}\right| = |1-0.9948| + |1-1.00332| + |1-1.000188| = 0.008708\,.$$

Hence, Gauss–Seidel seems to converge faster.

Example 2 *The linear system*

$$\begin{aligned}
2x_1 + 5x_2 + 5x_3 &= 12\,,\\
5x_1 + 2x_2 + 5x_3 &= 12\,,\\
5x_1 + 5x_2 + 2x_3 &= 12
\end{aligned} \tag{2}$$

has the unique solution $\mathbf{x}$ with entries $x_1 = x_2 = x_3 = 1$.

The coefficient matrix of the linear system (2) is

$$A = \begin{pmatrix} 2 & 5 & 5\\ 5 & 2 & 5\\ 5 & 5 & 2 \end{pmatrix}\,,$$

and we are given its eigenvalues $\lambda_1 = 12$, $\lambda_2 = \lambda_3 = -3$. *We denote by* $\mathbf{v}_i$ *the corresponding eigenvector to the eigenvalue* λ_i.

The Jacobi iteration matrix is given by $T = I - \frac{1}{2}A$ *and hence its eigenvalues are obtained from*

$$T\mathbf{v}_i = \left(I - \frac{1}{2}A\right)\mathbf{v}_i = \mathbf{v}_i - \frac{1}{2}\lambda_i\mathbf{v}_i = \left(1 - \frac{1}{2}\lambda_i\right)\mathbf{v}_i\,.$$

So the eigenvalues of T *are* $\mu_1 = 1 - \frac{1}{2}\lambda_1 = -5$, $\mu_2 = 1 - \frac{1}{2}\lambda_2 = \frac{5}{2}$, $\mu_3 = 1 - \frac{1}{2}\lambda_3 = \frac{5}{2}$. *Thus,* $\rho(T) = |\mu_1| = 5 > 1$ *and hence Jacobi will not converge for this matrix.*

Starting from $\mathbf{x}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, *the Jacobi iteration reads*

$$\begin{aligned} x_1^{(k+1)} &= \frac{12 - 5x_2^{(k)} - 5x_3^{(k)}}{2}\,, \\ x_2^{(k+1)} &= \frac{12 - 5x_1^{(k)} - 5x_3^{(k)}}{2}\,, \\ x_3^{(k+1)} &= \frac{12 - 5x_1^{(k)} - 5x_2^{(k)}}{2}\,, \end{aligned}$$

for $k = 0, 1, \ldots$. *We perform two iterations and obtain*

$$\begin{aligned} x_1^{(1)} &= 6\,, \qquad & x_1^{(2)} &= -24 \\ x_2^{(1)} &= 6\,, \qquad & x_2^{(2)} &= -24 \\ x_3^{(1)} &= 6\,, \qquad & x_3^{(2)} &= -24 \end{aligned}$$

As expected, Jacobi does not appear to converge.

Example 3 *The linear system*

$$\begin{aligned} 2x_1 - x_2 &= 3\,, \\ -x_1 + 2x_2 &= 0 \end{aligned} \tag{3}$$

has the unique solution $\mathbf{x}$ *with entries* $x_1 = 2$ *and* $x_2 = 1$.

The coefficient matrix of the linear system (3) is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The Jacobi iteration matrix is given by

$$T = I - \frac{1}{2}A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$$

In order to determine the eigenvalues of T, we need to solve for $\mathbf{v} \neq \mathbf{0}$

$$T\mathbf{v} = \mu\mathbf{v} \quad \Leftrightarrow \quad (T - \mu I)\,\mathbf{v} = \mathbf{0}\,,$$

and hence $\det(T - \mu I) = 0$. *In Chapter 3 (Review Linear Algebra), we saw how to calculate the determinant of a* 2×2 *matrix:*

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\,.$$

Hence

$$\det(T - \mu I) = \det\begin{pmatrix} -\mu & 1/2 \\ 1/2 & -\mu \end{pmatrix} = \mu^2 - \frac{1}{4}\,,$$

so we need to solve $\mu^2 - \frac{1}{4} = 0$, *which gives* $\mu_{1,2} = \pm\frac{1}{2}$. *Thus,* $\rho(T) = |\mu_1| = 0.5 < 1$, *and hence Jacobi will converge for* A *regardless of the starting vector* $\mathbf{x}^{(0)}$.

The Gauss–Seidel iteration matrix is given by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/4 \end{pmatrix}.$$

In order to determine the eigenvalues of T, we need to solve $\det(T - \mu I) = 0$. *Hence*

$$\det(T - \mu I) = \det\begin{pmatrix} -\mu & 1/2 \\ 0 & 1/4 - \mu \end{pmatrix} = -\mu\left(\frac{1}{4} - \mu\right),$$

so we need to solve $-\mu\left(\frac{1}{4} - \mu\right) = 0$, *which gives* $\mu_1 = 0$ *and* $\mu_2 = \frac{1}{4}$. *Thus,* $\rho(T) = |\mu_2| = 0.25 < 1$, *and hence Gauss–Seidel will converge for* A *regardless of the starting vector* $\mathbf{x}^{(0)}$.

The spectral radius of T for Gauss–Seidel is squared that of T of Jacobi, so Gauss–Seidel will converge twice as fast.