Uncertainty: wrap up & Decision Theory: Intro

CPSC 322 – Decision Theory 1

Textbook §6.4.1 & §9.2

March 30, 2011

Remarks on Assignment 4

- Question 2 (Bayesian networks)
  - "correctly represent the situation described above" means "do not make any independence assumptions that aren't true"
    - Step 1: identify the causal network
    - Step 2: for each network, check if it entails (conditional or marginal) independencies the causal network does not entail. If so, it's incorrect
  - Failing to entail some (or all) independencies does not make a network incorrect (only computationally suboptimal)

- Question 5 (Rainbow Robot)
  - If you got rainbowrobot.zip before Sunday, get the updated version: rainbowrobot_updated.zip (on WebCT)

- Question 4 (Decision Networks)
  - This is mostly Bayes rule and common sense
  - One could compute the answer algorithmically, but you don't need to

Lecture Overview

- Variable elimination: recap and some more details
- Variable elimination: pruning irrelevant variables
- Summary of Reasoning under Uncertainty
- Decision Theory
  - Intro
  - Time-permitting: Single-Stage Decision Problems

Recap: Factors and Operations on them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., assign X=t

Factor of Y,X,Z

X	Y	Z	$f_1(X,Y,Z)$
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
~~f~~	~~t~~	~~t~~	~~0.4~~
~~f~~	~~t~~	~~f~~	~~0.6~~
~~f~~	~~f~~	~~t~~	~~0.3~~
~~f~~	~~f~~	~~f~~	~~0.7~~

$$\boxed{f_1(X,Y,Z)_{X=t} = f_2(Y,Z)}$$

Y	Z	$f_2(Y,Z)$
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

Factor of Y,Z

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor

$$\boxed{\sum_B f_3(A,B,C) = f_4(A,C)}$$

B	A	C	$f_3(A,B,C)$
t	t	t	0.03
t	t	f	0.07
f	t	t	0.54
f	t	f	0.36
t	f	t	0.06
t	f	f	0.14
f	f	t	0.48
f	f	f	0.32

A	C	$f_4(A,C)$
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Recap: Operation 3: multiplying factors

A	B	$f_5(A,B)$
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$f_5(A,B) \times f_6(B,C) = f_7(A,B,C)$, i.e

$f_5(A=a,B=b) \times f_6(B=b,C=c) = f_7(A=a,B=b,C=c)$

A	C	$f_6(A,C)$
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

A	B	C	$f_7(A,B,C)$
t	t	t	0.03
t	t	f	0.1x0.7
t	f	t	0.9x0.6
t	f	f	…
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- If we assign variable A=a in factor $f_7(A,B)$, what is the correct form for the resulting factor?

f(A) | f(B) | f(A,B) | A number

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- If we assign variable A=a in factor $f_7(A,B)$, what is the correct form for the resulting factor?
  - f(B).
    When we assign variable A we remove it from the factor's domain

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A{=}a,B{=}b,C{=}c) = f_5(A{=}a,B{=}b) \times f_6(B{=}b,C{=}c)$

- If we marginalize variable A out from factor $f_7(A,B)$, what is the correct form for the resulting factor?

f(A) | f(B) | f(A,B) | A number

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- If we assign variable A=a in factor $f_7(A,B)$, what is the correct form for the resulting factor?
  - f(B).
    When we marginalize out variable A we remove it from the factor's domain

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- If we multiply factors $f_4(X,Y)$ and $f_6(Z,Y)$, what is the correct form for the resulting factor?

f(X,Z) | f(X) | f(X,Y) | f(X,Y,Z)

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- If we multiply factors $f_4(X,Y)$ and $f_6(Z,Y)$, what is the correct form for the resulting factor?
  - $f(X,Y,Z)$
  - When multiplying factors, the resulting factor's domain is the union of the multiplicands' domains

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- What is the correct form for $\sum_B f_5(A,B) \times f_6(B,C)$
  - As usual, product before sum: $\sum_B (\; f_5(A,B) \times f_6(B,C) \quad)$

f(A,B,C) f(B) f(A,C) f(B,C)

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$

- What is the correct form for $\sum_B f_5(A,B) \times f_6(B,C)$
  - As usual, product before sum: $\sum_B (\ f_5(A,B) \times f_6(B,C) \ )$
  - Result of multiplication: f(A,B,C). Then marginalize out B: f'(A,C)

Recap: Factors and Operations on Them

- A factor is a function from a tuple of random variables to the real numbers $\mathbb{R}$
- Operation 1: assigning a variable in a factor
  - E.g., $f_2(Y,Z) = f_1(X,Y,Z)_{X=t}$
- Operation 2: marginalize out a variable from a factor
  - E.g., $f_4(A,C) = \sum_B f_3(A,B,C)$
- Operation 3: multiply two factors
  - E.g. $f_7(A,B,C) = f_5(A,B) \times f_6(B,C)$
    - That means, $f_7(A=a,B=b,C=c) = f_5(A=a,B=b) \times f_6(B=b,C=c)$
- Operation 4: normalize the factor
  - Divide each entry by the sum of the entries. The result will sum to 1.

A	$f_5(A,B)$
t	0.4
f	0.1

A	$f_6(A,B)$
t	0.4/(0.4+0.1) = 0.8
f	0.1/(0.4+0.1) = 0.2

Recap: the Key Idea of Variable Elimination

- An efficient way to sum out a variable Z_k from a product $f_1 \times \ldots \times f_k$ of factors:
  - Partition the factors into
    - those that don't contain Z_k, say $f_1 \times \ldots \times f_i$
    - those that contain Z_k, say $f_{i+1} \times \ldots \times f_k$
- Since multiplication distributes over addition:

$$\sum_{Z_k} f_1 \times \ldots \times f_k = f_1 \times \ldots \times f_i \times \left(\sum_{Z_k} f_{i+1} \times \ldots \times f_k \right)$$

New factor! Let's call it f'

- Store f' explicitly, and discard $f_{i+1} \ldots f_k$
- Now we've summed out Z_k

Recap: Variable Elimination (VE) in BNs

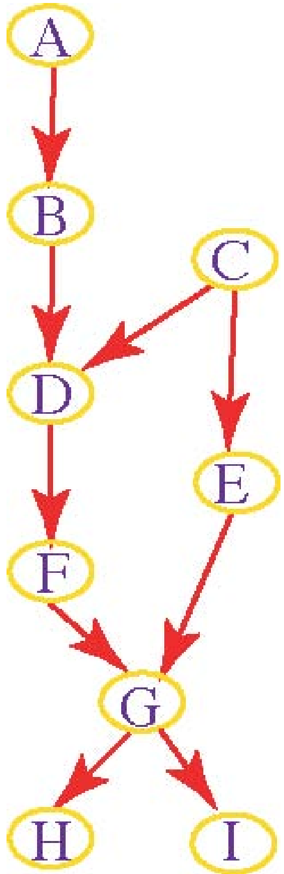
- The joint probability distribution of a Bayesian network is $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i|pa(X_i))$
  - We make a factor f_i for each conditional probability table $P(X_i|pa(X_i))$
  - So we have $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} f_i$

- The variable elimination algorithm computes $P(Y|\, E_1=e_1, \ldots, E_j=e_j)$ as follows:
  - Assign $E_1=e_1, \ldots, E_j=e_j$
  - Sum out all non-query variables $Z_1, \ldots, Z_k$, one at a time
    - To sum out Z_i:
      - Multiply factors containing it Z_i
      - Then marginalize out Z_i from the product
    - The order in which we sum out variables is called our elimination ordering
  - Normalize the final factor $f(Y)$.
    - The resulting factor is exactly $P(Y|\, E_1=e_1, \ldots, E_j=e_j)$

Recap: VE example: compute $P(G|H=h_1)$
Step 1: construct a factor for each cond. probability

$P(G,H) =$

$$\sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_7(H,G)\, f_8(I,G)$$

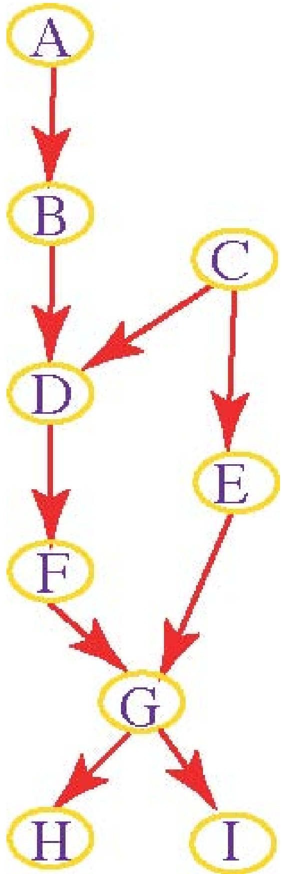


Recap: VE example: compute $P(G|H=h_1)$

Step 2: assign observed variables their observed value

$P(G,H) =$

$$\sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_7(H,G)\, f_8(I,G)$$



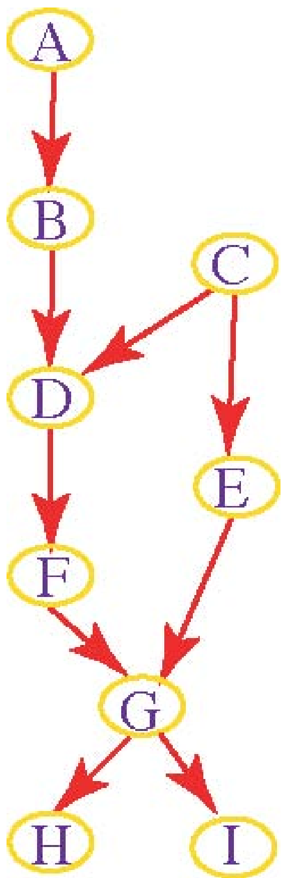
Assigning the variable $H=h_1$:

$f_9(G) = f_7(H,G)_{H=h1}$

$$P(G,H=h_1)=\sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

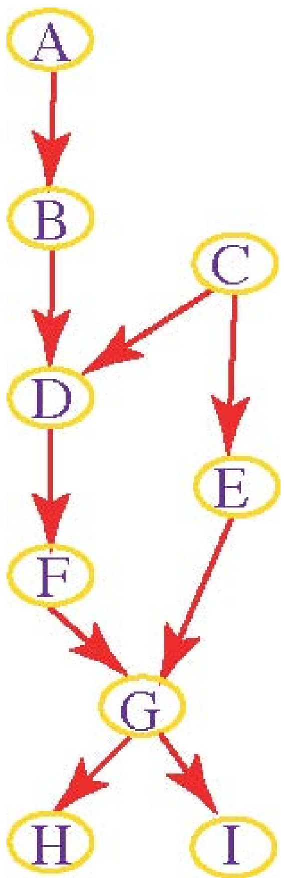


Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$



Summing out variable *A:*

$\sum_A f_0(A)\, f_1(B,A) = f_{10}(B)$

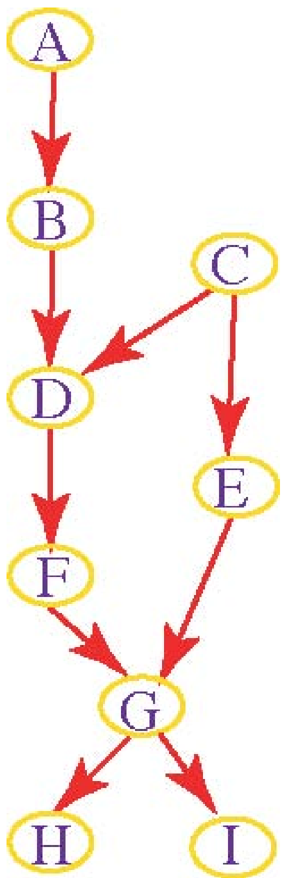
Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$

$= \sum_{B,D,E,F,I} f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$



Summing out variable *C:*

$\sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) = f_{11}(B,D,E)$

Elimination ordering: A, C, E, I, B, D, F

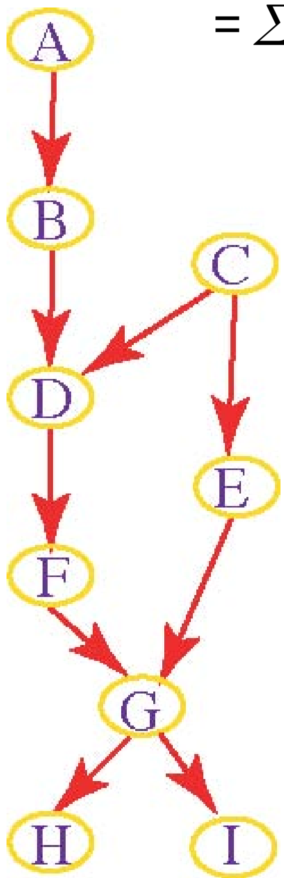
Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$

$= \sum_{B,D,E,F,I} f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$

$= \sum_{B,D,F,I} f_5(F, D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$



Summing out variable *E:*

$\sum_E f_6(G,F,E)\, f_{11}(B,D,E) = f_{12}(G,F,B,D)$

Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

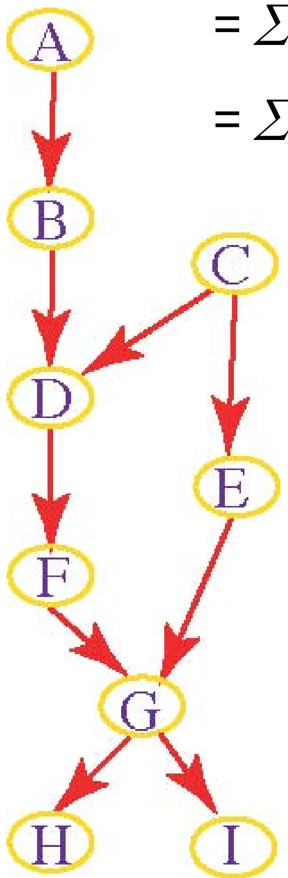
$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

$$= \sum_{B,D,E,F,I} f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

$$= \sum_{B,D,F,I} f_5(F,D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

$$= \sum_{B,D,F} f_5(F,D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$



Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$

Step 4: sum out non- query variables (one at a time)

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

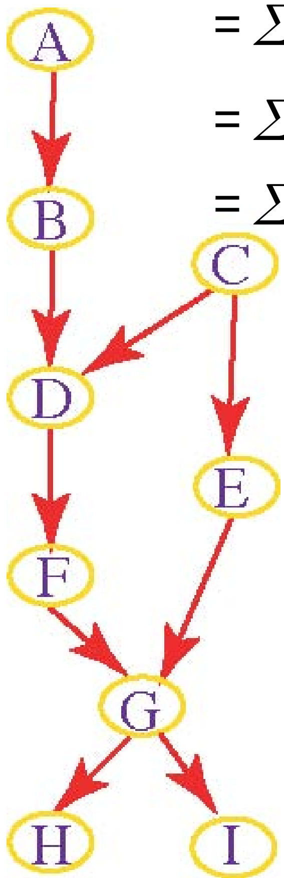
$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

$$= \sum_{B,D,E,F,I} f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

$$= \sum_{B,D,F,I} f_5(F,D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

$$= \sum_{B,D,F} f_5(F,D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$

$$= \sum_{D,F} f_5(F,D)\, f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{14}(G,F,D)$$



Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$

Step 4: sum out non- query variables (one at a time)

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

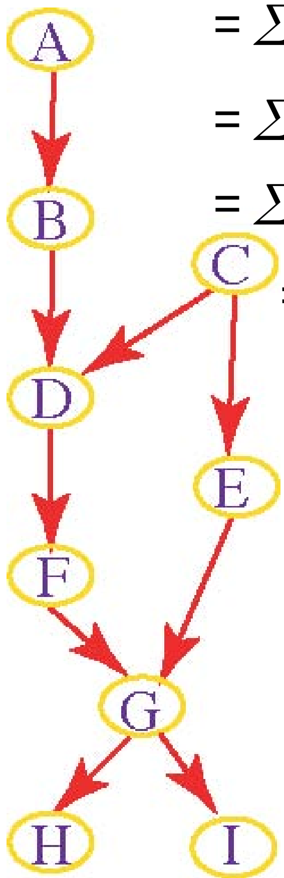
$$= \sum_{B,D,E,F,I} f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

$$= \sum_{B,D,F,I} f_5(F, D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

$$= \sum_{B,D,F} f_5(F, D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$

$$= \sum_{D,F} f_5(F, D)\, f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{14}(G,F,D)$$

$$= \sum_{F} f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{15}(G,F)$$



Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

$$= \sum_{B,D,E,F,I} f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

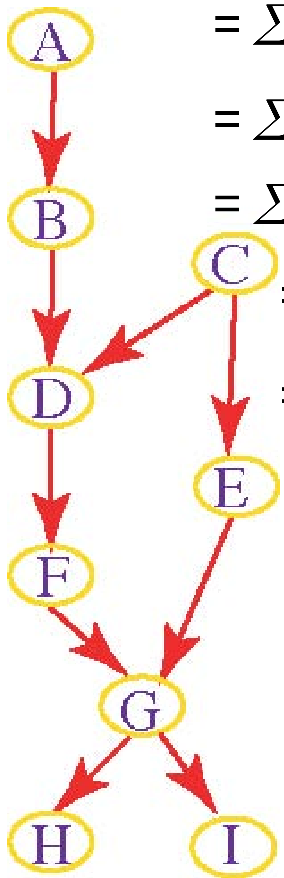
$$= \sum_{B,D,F,I} f_5(F,D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

$$= \sum_{B,D,F} f_5(F,D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$

$$= \sum_{D,F} f_5(F,D)\, f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{14}(G,F,D)$$

$$= \sum_{F} f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{15}(G,F)$$

$$= f_9(G)\, f_{12}(G)\, f_{16}(G)$$



Elimination ordering: A, C, E, I, B, D, F

Recap: VE example: compute $P(G|H=h_1)$
Step 5: multiply the remaining factors

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

$$= \sum_{B,D,E,F,I} f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

$$= \sum_{B,D,F,I} f_5(F, D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

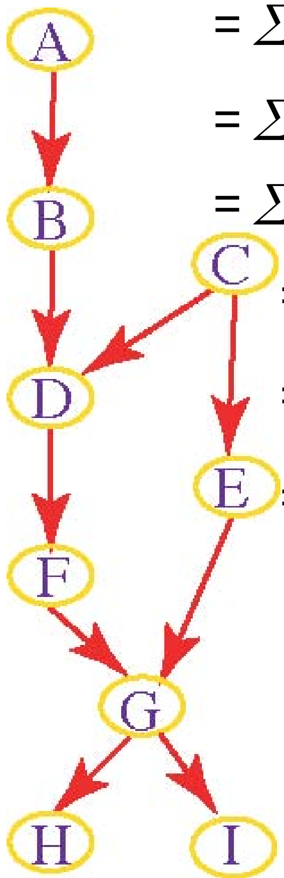
$$= \sum_{B,D,F} f_5(F, D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$

$$= \sum_{D,F} f_5(F, D)\, f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{14}(G,F,D)$$

$$= \sum_{F} f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{15}(G,F)$$

$$= f_9(G)\, f_{12}(G)\, f_{16}(G)$$

$$= f_{17}(G)$$



Recap: VE example: compute $P(G|H=h_1)$
Step 6: normalize

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

$$= \sum_{B,C,D,E,F,I} f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)$$

$$= \sum_{B,D,E,F,I} f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{11}(B,D,E)$$

$$= \sum_{B,D,F,I} f_5(F, D)\, f_9(G)\, f_8(I,G)\, f_{10}(B)\, f_{12}(G,F,B,D)$$

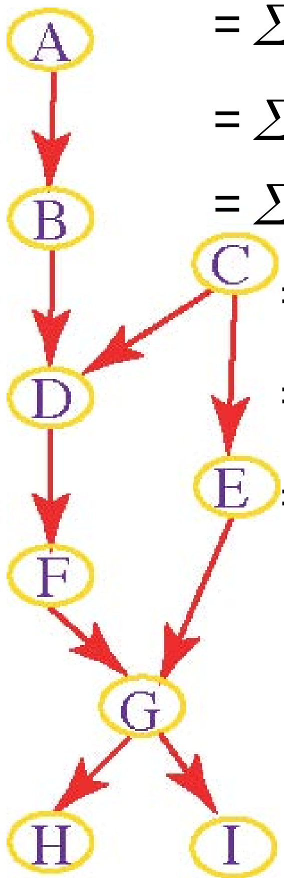
$$= \sum_{B,D,F} f_5(F, D)\, f_9(G)\, f_{10}(B)\, f_{12}(G,F,B,D)\, f_{13}(G)$$

$$= \sum_{D,F} f_5(F, D)\, f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{14}(G,F,D)$$

$$= \sum_{F} f_9(G)\, f_{11}(G,F)\, f_{12}(G)\, f_{15}(G,F)$$

$$= f_9(G)\, f_{12}(G)\, f_{16}(G)$$

$$= f_{17}(G)$$



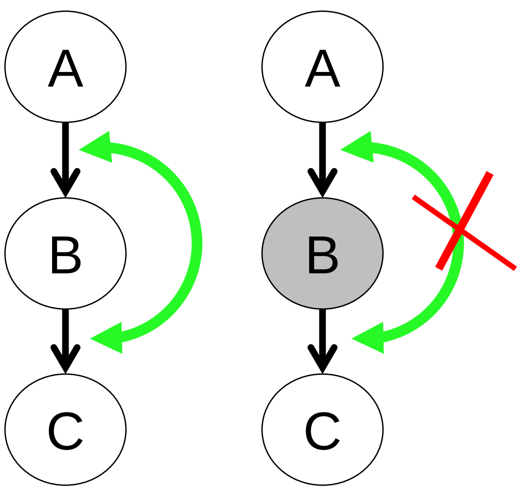
$$\boxed{P(G=g \mid H=h_1) = \frac{f_{17}(g)}{\sum_{g' \in dom(G)} f_{17}(g')}}$$

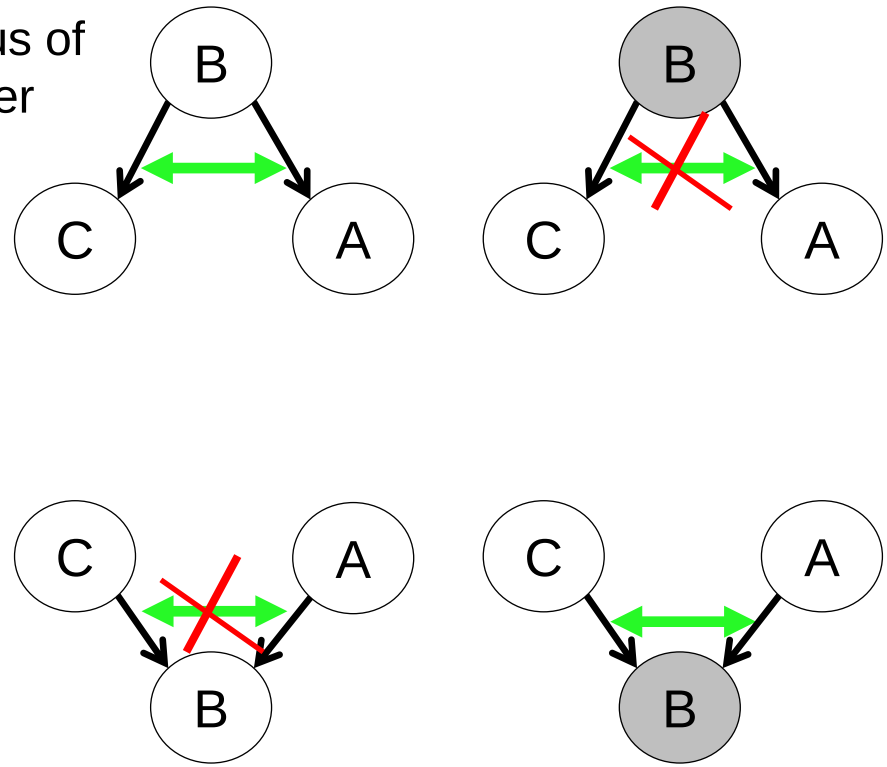
Lecture Overview

- Variable elimination: recap and some more details
- Variable elimination: pruning irrelevant variables
- Summary of Reasoning under Uncertainty
- Decision Theory
  - Intro
  - Time-permitting: Single-Stage Decision Problems

Recap: conditional independence in BNs

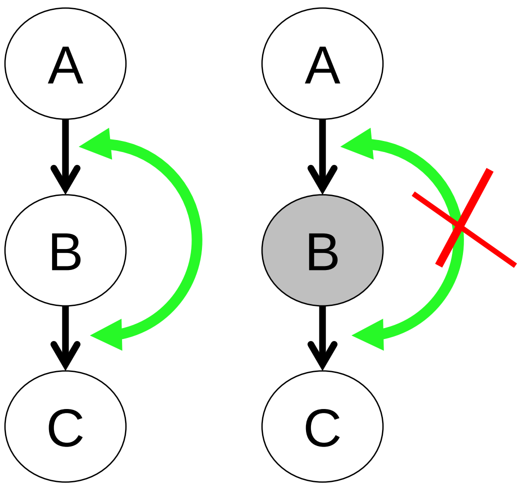
- Two variables X and Y are conditionally independent given a set of observed variables E, if and only if
  - There is no path along which information can flow from X to Y
  - Information can flow along a path if it can flow through all the nodes in the path.
    - Note: observation status of A and C does not matter





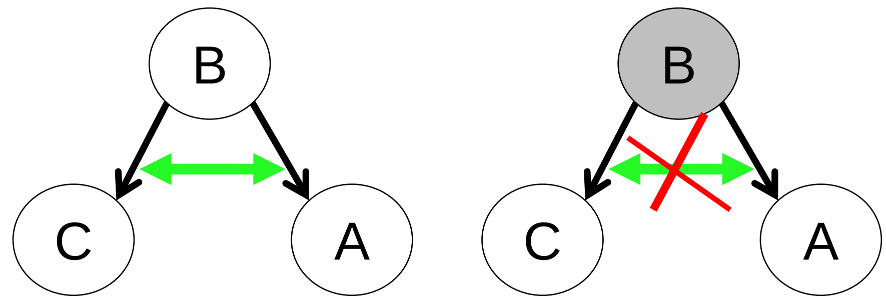
Conditional independence in BNs

- Memoization trick:
  - Assume that whether kids are nice
    depends only on whether their parents are nice
  - Assume that people get married independent of their niceness
  - Then “child” in a Bayesian network translates to child in the real world

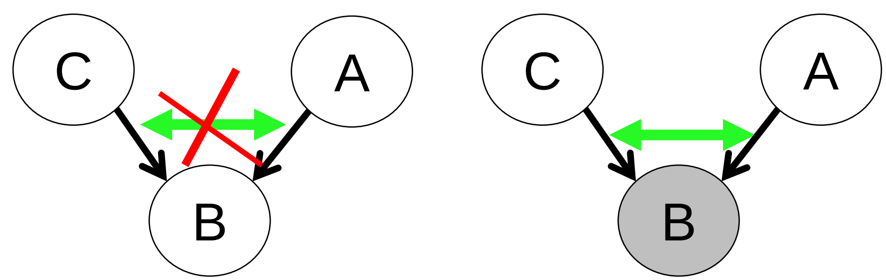


Your grandparent is nice, so your parent is likely to be nice, so you are likely to be nice.

But if we know how nice your parent is, the grandparent’s niceness doesn’t provide extra information.



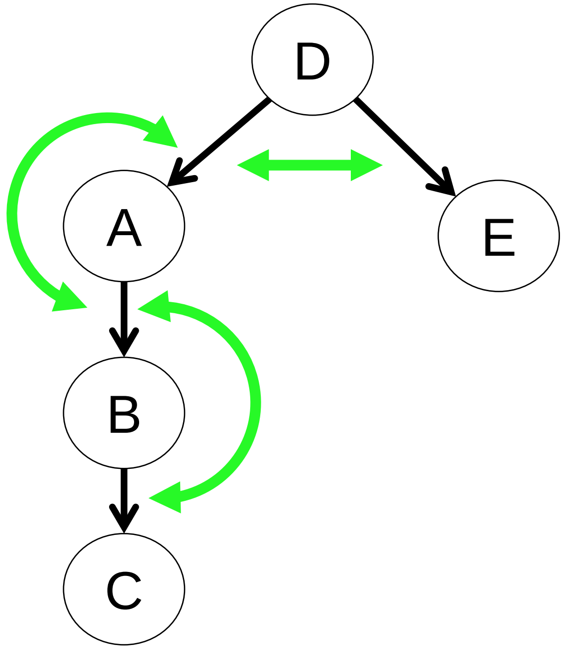
Nice people are likely to have nice sibblings since they have the same parent. But if you know the parent’s niceness, then that explains everything.



The dad is nice, that tells us nothing about the mom.
But if we know the kid is mean, the mom is likely mean.

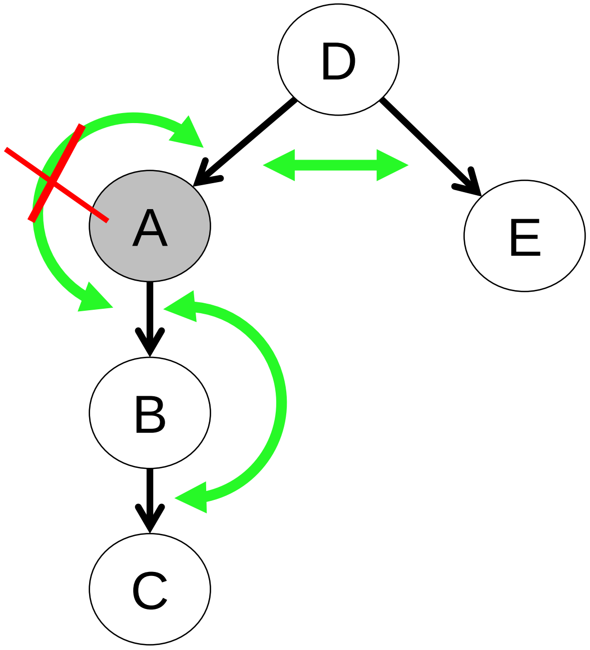
Conditional independence in BNs example

- Is E marginally independent of C?
  - No. Information flows between them (through all nodes on the path).



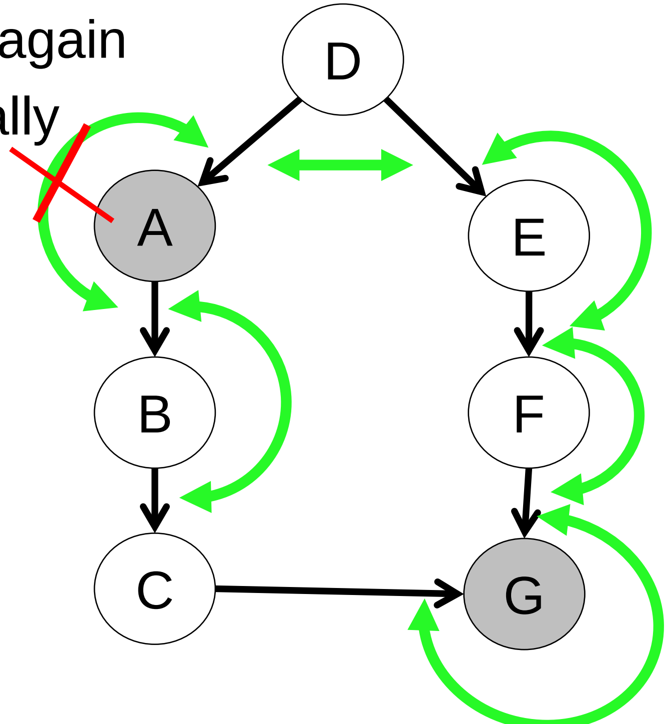
Conditional independence in BNs example

- Is E marginally independent of C?
  - No. Information flows between them (through all nodes on the path).
- What if we observe A?
  - I.e., is E conditionally independent of C given A?
  - Yes. The observed node in a chain blocks information.



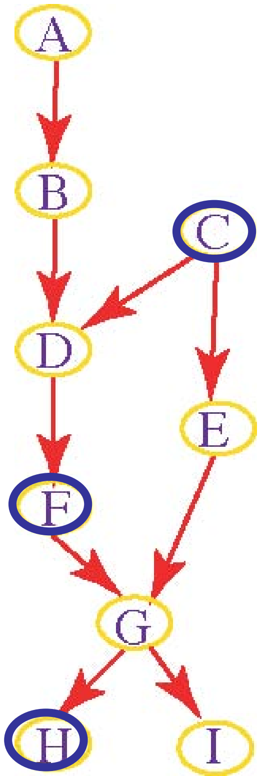
Conditional independence in BNs example

- Is E marginally independent of C?
  - No. Information flows between them (through all nodes on the path).
- What if we observe A?
  - I.e., is E conditionally independent of C given A?
  - Yes. The observed node in a chain blocks information.
- What if we add nodes F and G (observed)?
  - Now the information can flow again
  - So E and C are not conditionally independent given G and A



VE and conditional independence

- So far, we haven't use conditional independence in VE!
  - Before running VE, we can prune all variables Z that are conditionally independent of the query Y given evidence E: $Z \perp\!\!\!\perp Y \mid E$

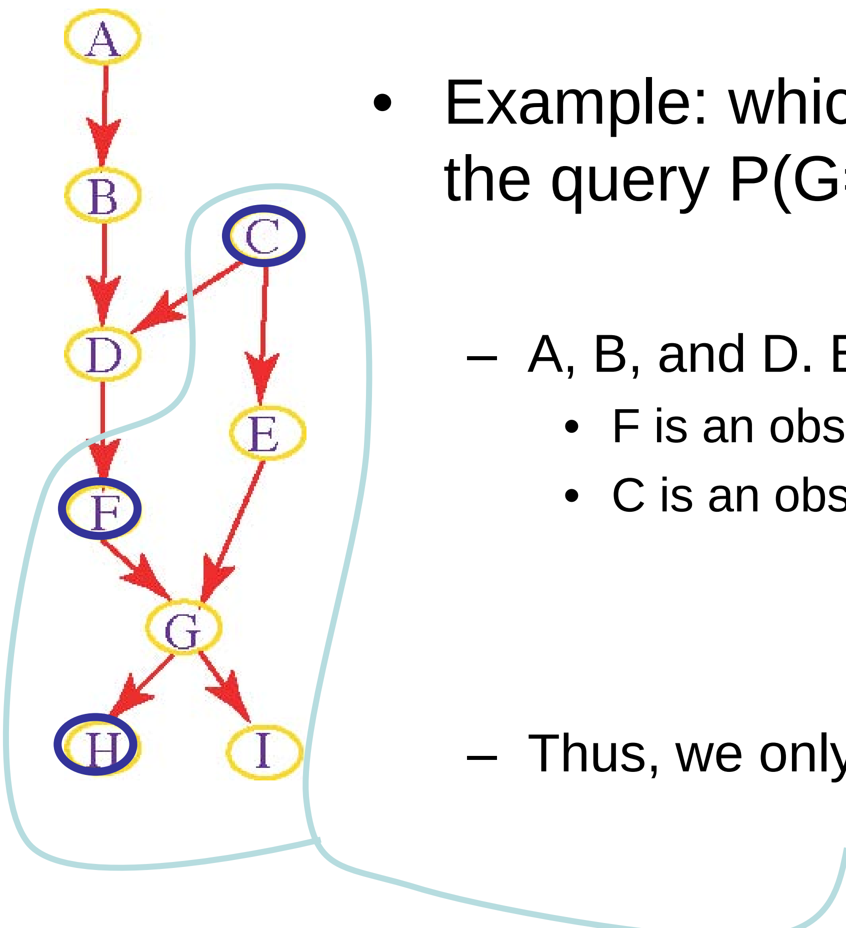


- Example: which variables can we prune for the query $P(G{=}g \mid C{=}c_1, F{=}f_1, H{=}h_1)$?

A B D E

VE and conditional independence

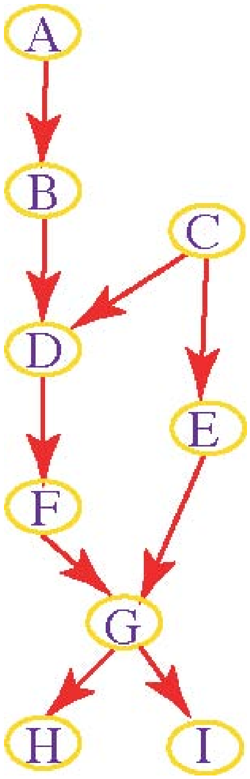
- So far, we haven't use conditional independence!
  - Before running VE, we can prune all variables Z that are conditionally independent of the query Y given evidence E: $Z \perp\!\!\!\perp Y \mid E$



- Example: which variables can we prune for the query $P(G{=}g \mid C{=}c_1, F{=}f_1, H{=}h_1)$?
  - A, B, and D. Both paths are blocked
    - F is an observed node in a chain structure
    - C is an observed common parent
  - Thus, we only need to consider this subnetwork

Variable Elimination: One last trick

- We can also prune unobserved leaf nodes
  - And we can do so recursively



E.g., which nodes can we prune if the query is P(A)?

H

I

G

All nodes other than A

Recursively prune unobserved leaf nodes:
we can prune all nodes other than A !

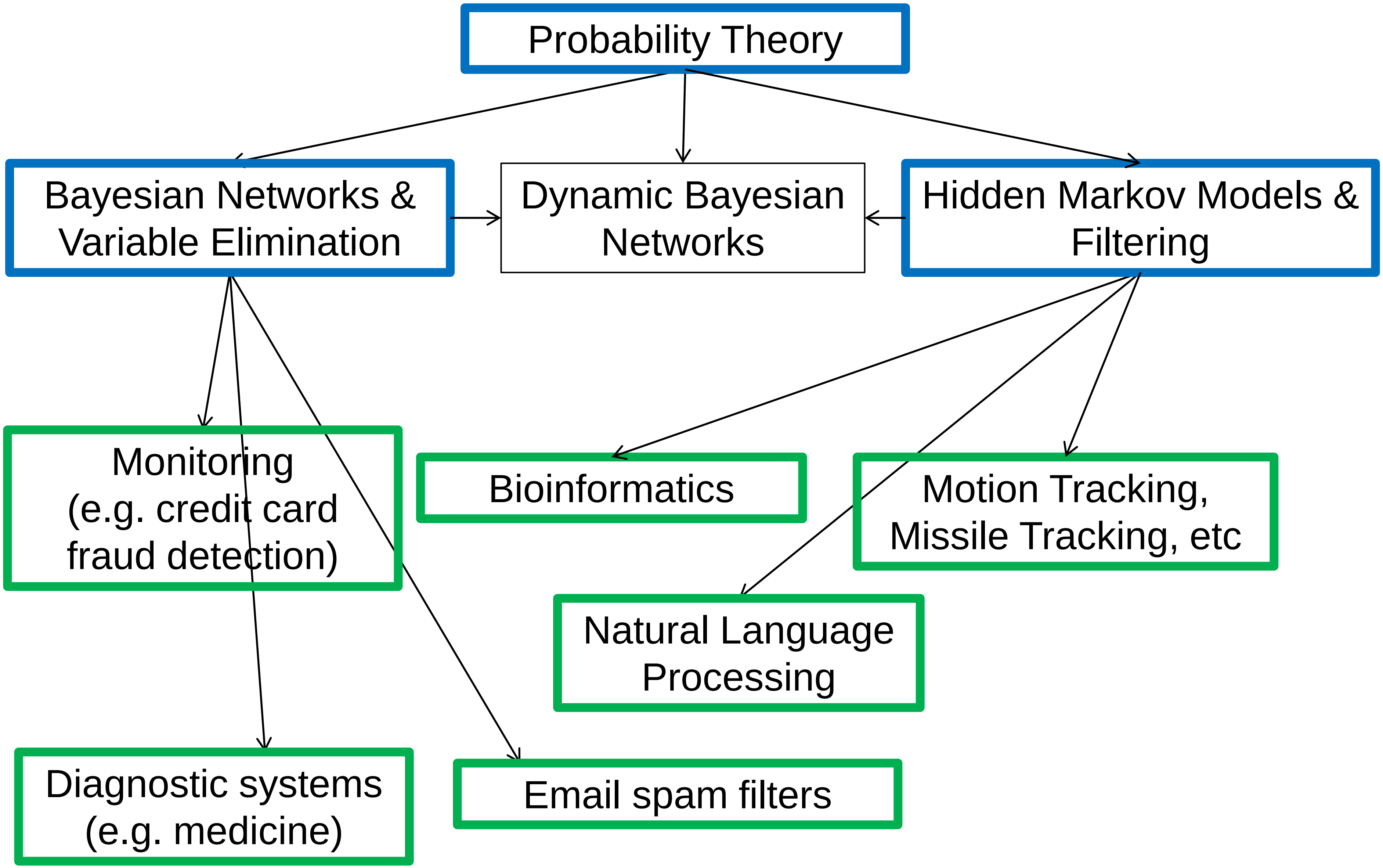
Complexity of Variable Elimination (VE)

- A factor over n binary variables has to store 2^n numbers
  - The initial factors are typically quite small (variables typically only have few parents in Bayesian networks)
  - But variable elimination constructs larger factors by multiplying factors together
- The complexity of VE is exponential in the maximum number of variables in any factor during its execution
  - This number is called the treewidth of a graph (along an ordering)
  - Elimination ordering influences treewidth
- Finding the best ordering is NP complete
  - I.e., the ordering that generates the minimum treewidth
  - Heuristics work well in practice (e.g. least connected variables first)
  - Even with best ordering, inference is sometimes infeasible
    - In those cases, we need approximate inference. See CS422 & CS540

Lecture Overview

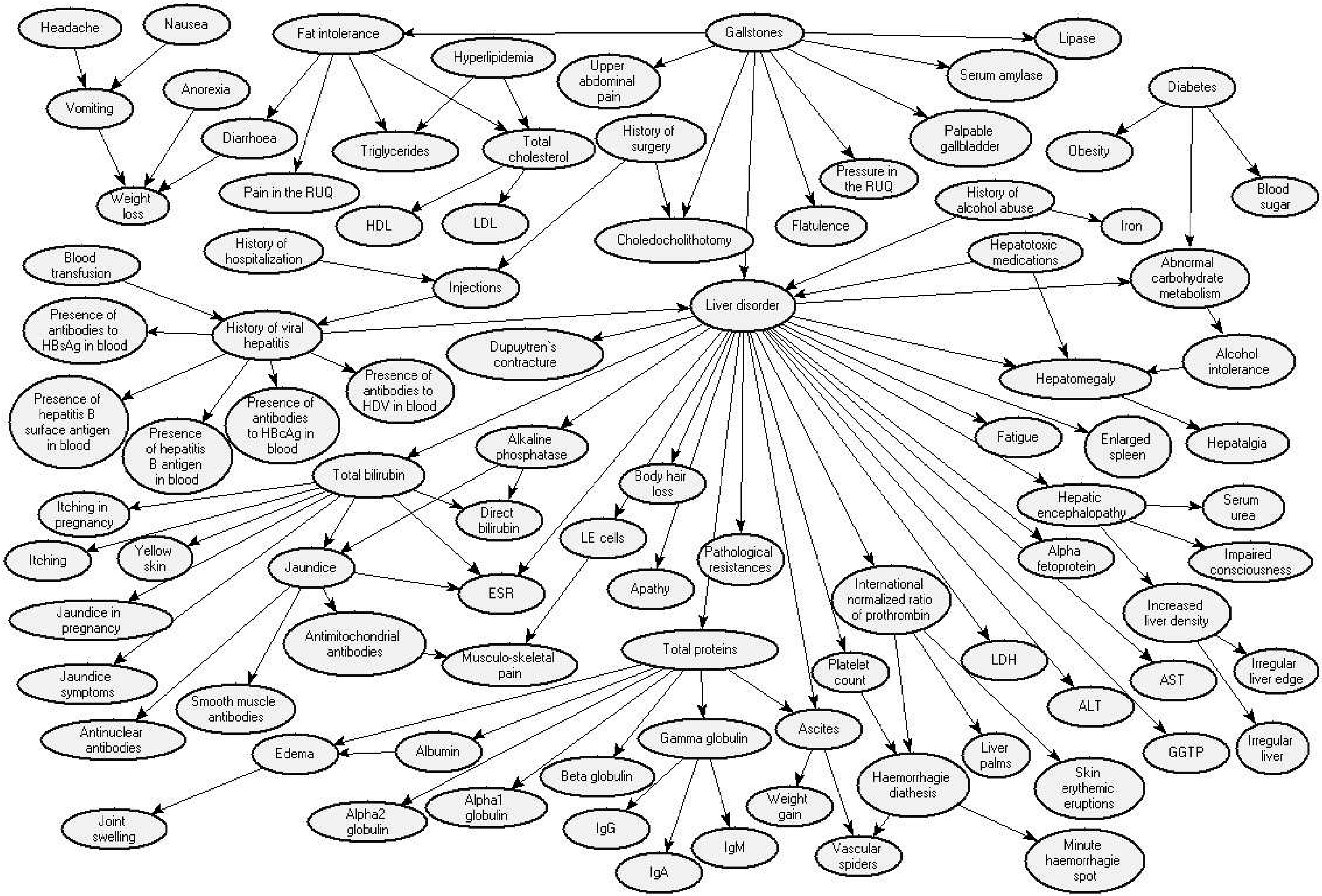
- Variable elimination: recap and some more details
- Variable elimination: pruning irrelevant variables
- Summary of Reasoning under Uncertainty
- Decision Theory
  - Intro
  - Time-permitting: Single-Stage Decision Problems

Big picture: Reasoning Under Uncertainty



One Realistic BNet: Liver Diagnosis

Source: Onisko et al., 1999



~60 nodes, max 4 parents per node

Course Overview

Course Module

Representation

Reasoning Technique

Environment

Problem Type

Problem Type	Course Module	Deterministic	Stochastic	Course Module
Static	Constraint Satisfaction	***Variables + Constraints*** Arc Consistency Search		
Static	Logic	***Logics*** Search	***Bayesian Networks*** Variable Elimination	Uncertainty
Sequential	Planning	***STRIPS*** Search As CSP (using arc consistency)	***Decision Networks*** Variable Elimination ***Markov Processes*** Value Iteration	Decision Theory

This concludes the uncertainty module

Course Overview

Course Module

Representation
Reasoning Technique

Environment

Problem Type		Deterministic	Stochastic	
Static	Constraint Satisfaction	*Variables + Constraints* Arc Consistency Search		
Static	Logic	*Logics* Search	*Bayesian Networks* Variable Elimination	Uncertainty
Sequential	Planning	*STRIPS* Search As CSP (using arc consistency)	*Decision Networks* Variable Elimination *Markov Processes* Value Iteration	Decision Theory

But uncertainty is also at the core of decision theory: now we're acting under uncertainty

Lecture Overview

- Variable elimination: recap and some more details
- Variable elimination: pruning irrelevant variables
- Summary of Reasoning under Uncertainty
- Decision Theory
  - Intro
  - Time-permitting: Single-Stage Decision Problems

Decisions Under Uncertainty: Intro

- Earlier in the course, we focused on decision making in deterministic domains
  - Search/CSPs: single-stage decisions
  - Planning: sequential decisions

- Now we face stochastic domains
  - so far we've considered how to represent and update beliefs
  - What if an agent has to make decisions under uncertainty?

- Making decisions under uncertainty is important
  - We mainly represent the world probabilistically so we can use our beliefs as the basis for making decisions

Decisions Under Uncertainty: Intro

- An agent's decision will depend on
  - What actions are available
  - What beliefs the agent has
  - Which goals the agent has

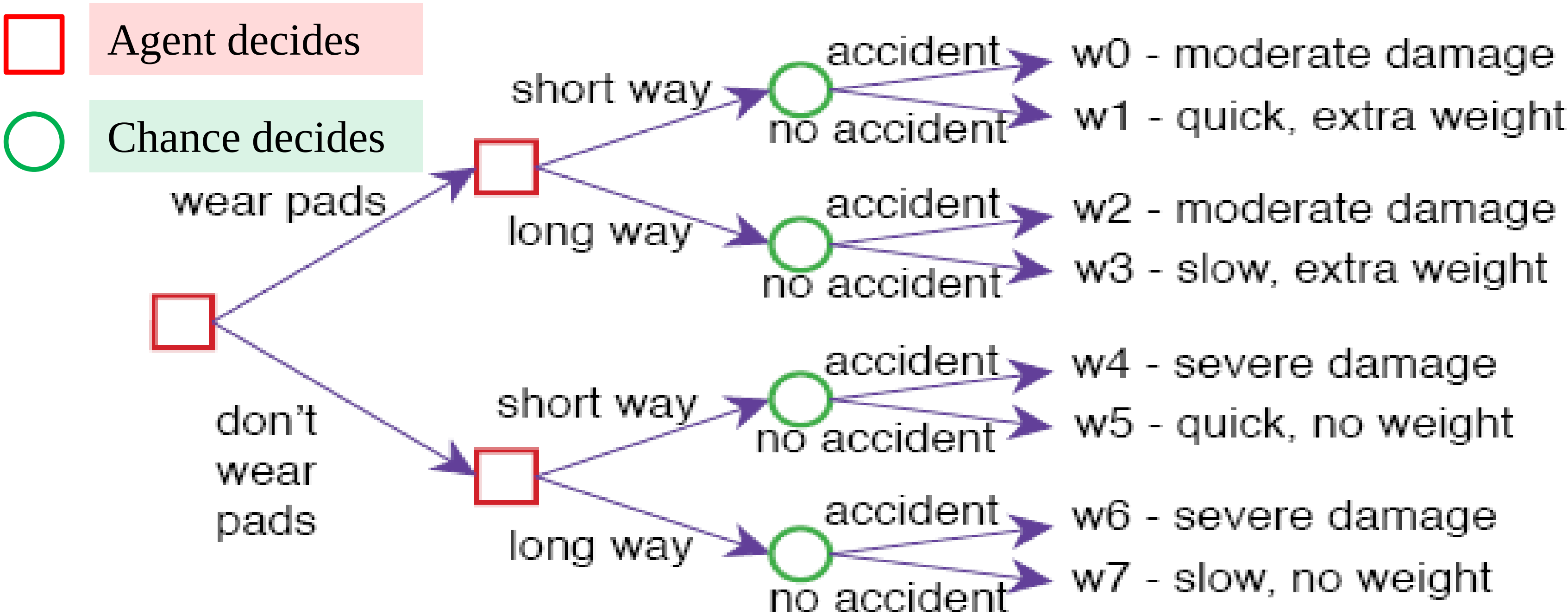
- Differences between deterministic and stochastic setting
  - Obvious difference in representation: need to represent our uncertain beliefs
  - Now we'll speak about representing actions and goals
    - Actions will be pretty straightforward: decision variables
    - Goals will be interesting: we'll move from all-or-nothing goals to a richer notion: rating how happy the agent is in different situations.
    - Putting these together, we'll extend Bayesian networks to make a new representation called decision networks

Lecture Overview

- Variable elimination: recap and some more details
- Variable elimination: pruning irrelevant variables
- Summary of Reasoning under Uncertainty
- Decision Theory
  - Intro
  - Time-permitting: Single-Stage Decision Problems

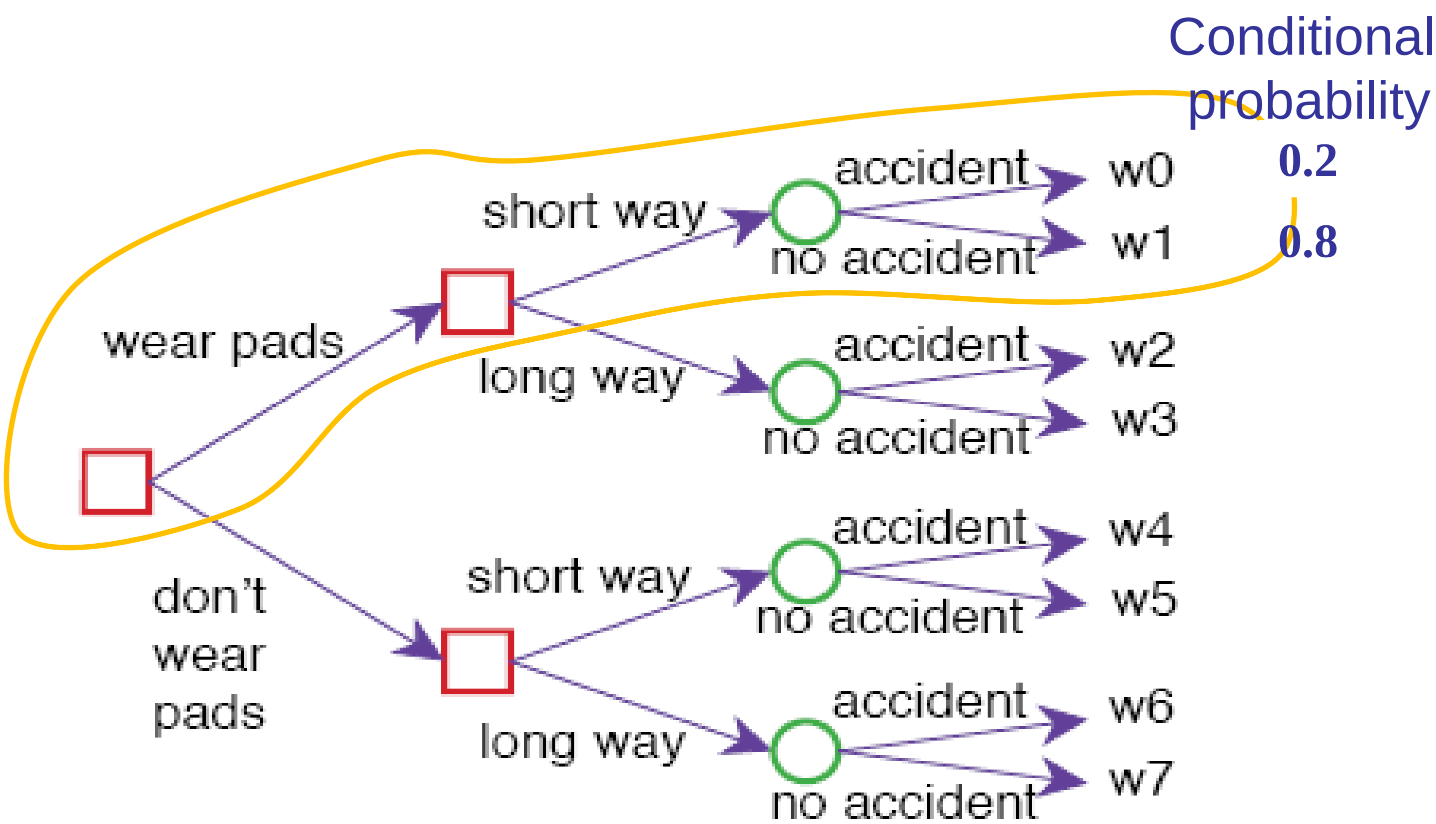
Delivery Robot Example

- Decision variable 1: the robot can choose to wear pads
  - Yes: protection against accidents, but extra weight
  - No: fast, but no protection
- Decision variable 2: the robot can choose the way
  - Short way: quick, but higher chance of accident
  - Long way: safe, but slow
- Random variable: is there an accident?



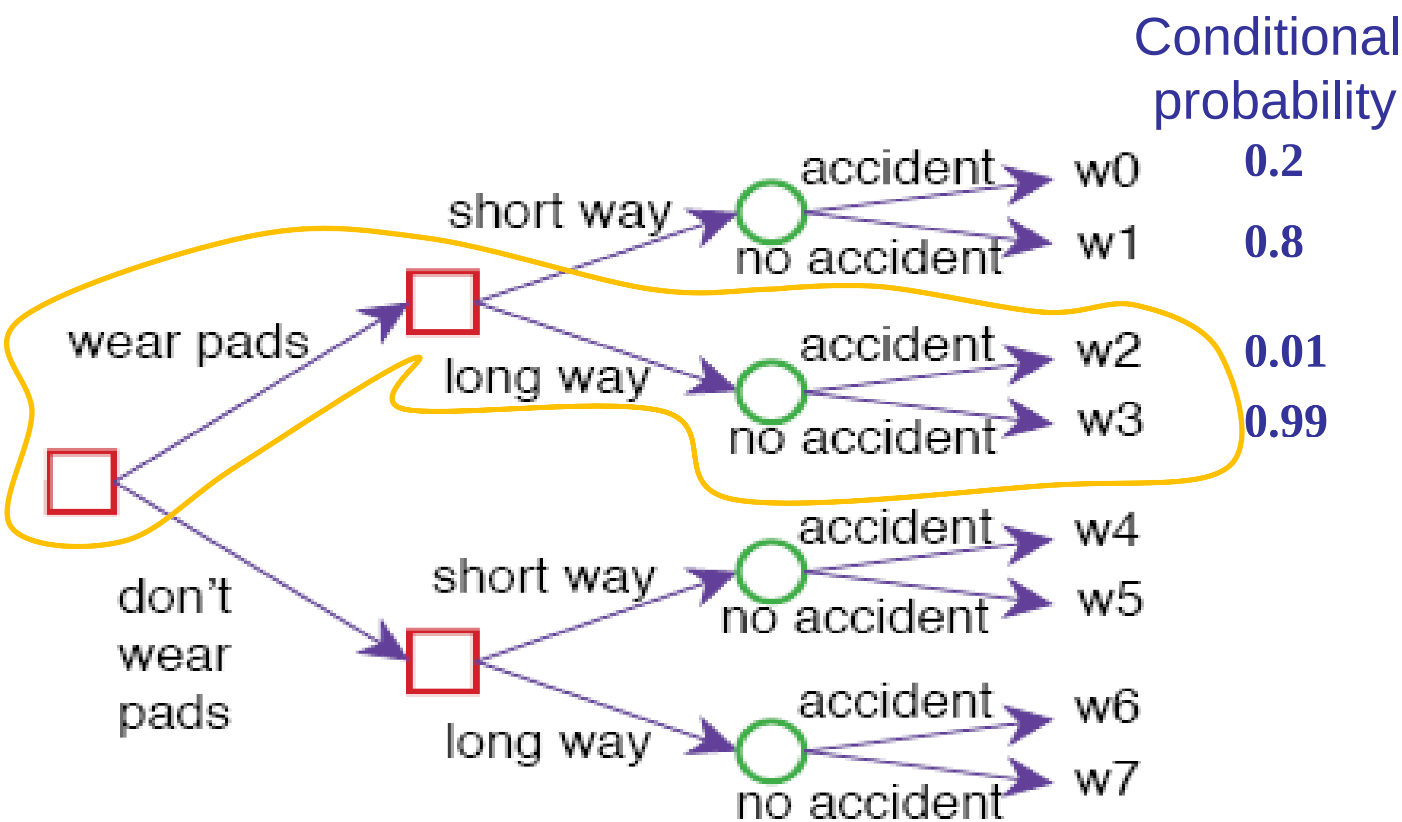
Possible worlds and decision variables

- A possible world specifies a value
  for each random variable and each decision variable
- For each assignment of values to all decision variables
  - the probabilities of the worlds satisfying that assignment sum to 1.



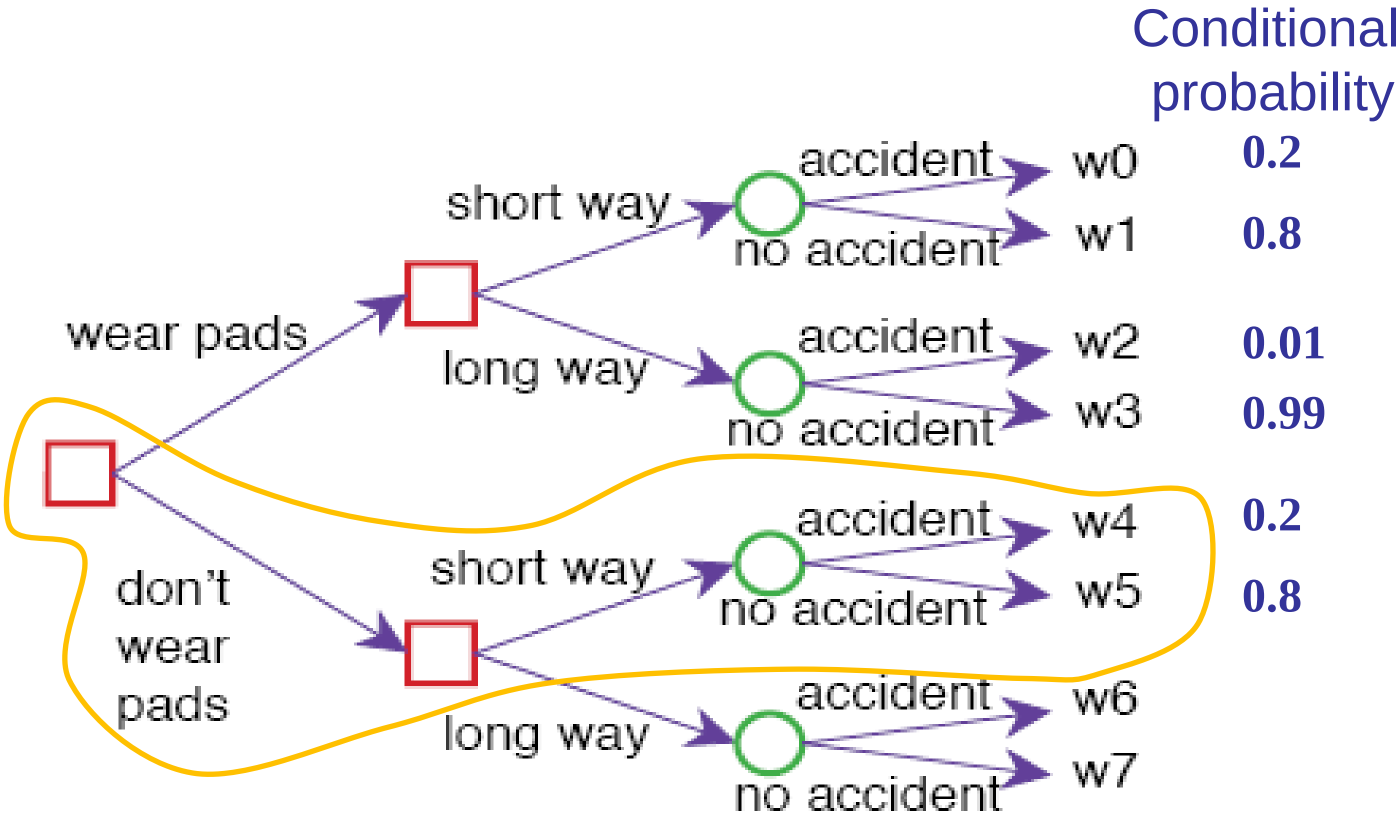
Possible worlds and decision variables

- A possible world specifies a value
  for each random variable and each decision variable
- For each assignment of values to all decision variables
  – the probabilities of the worlds satisfying that assignment sum to 1.



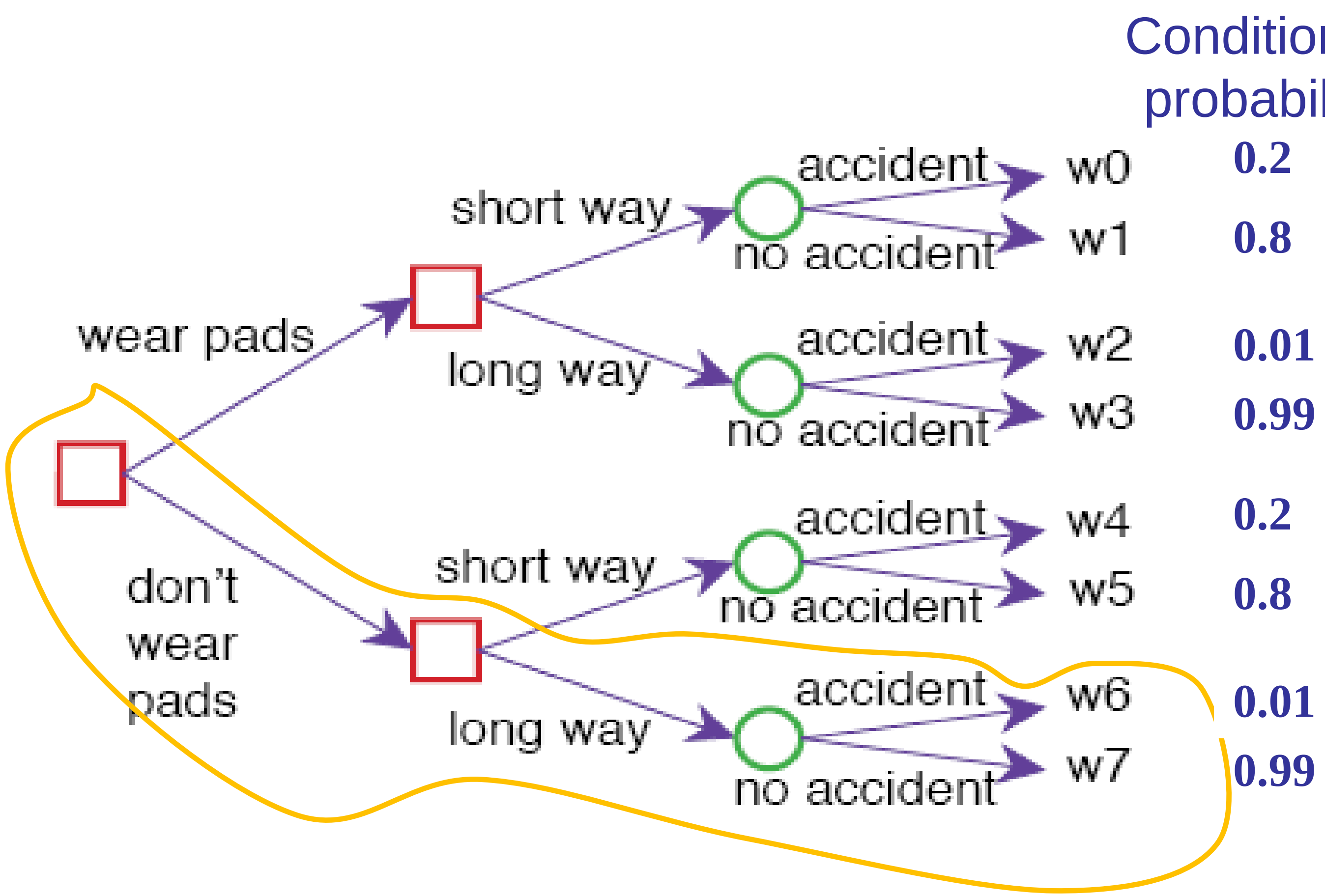
Possible worlds and decision variables

- A possible world specifies a value
  for each random variable and each decision variable
- For each assignment of values to all decision variables
  – the probabilities of the worlds satisfying that assignment sum to 1.



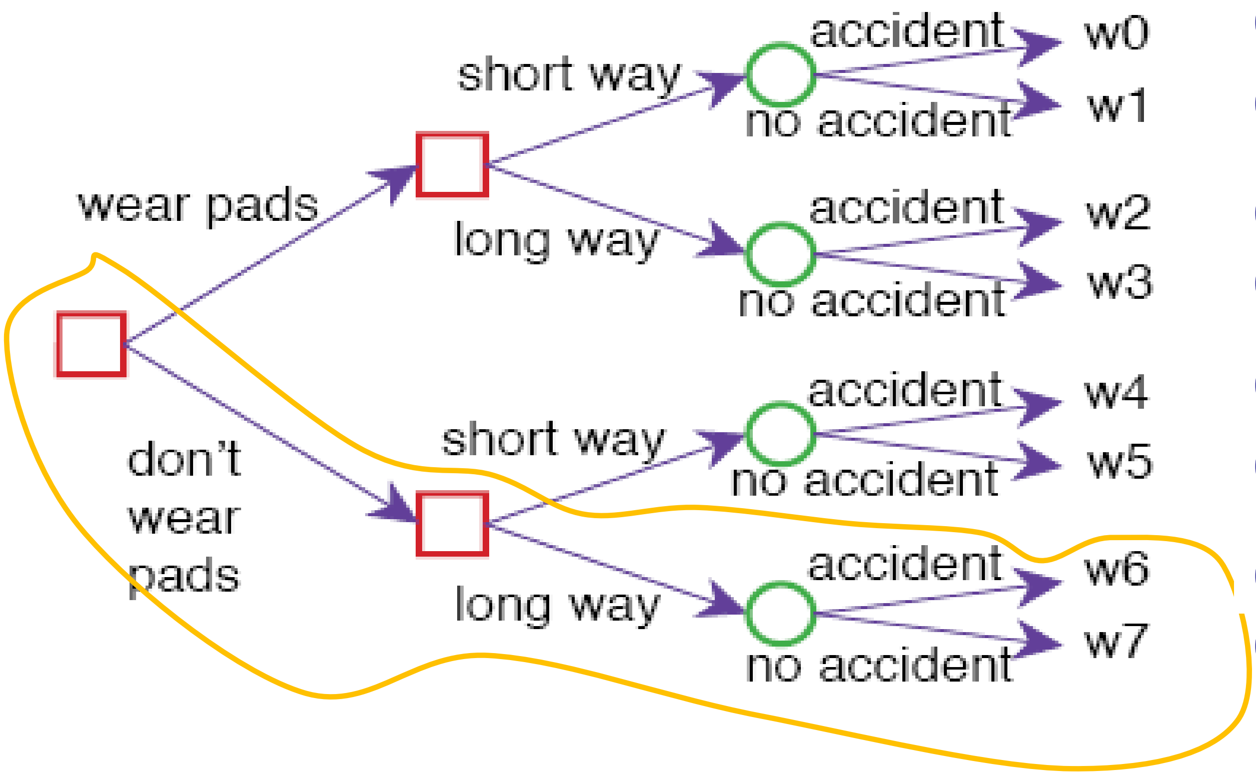
Possible worlds and decision variables

- A possible world specifies a value for each random variable and each decision variable
- For each assignment of values to all decision variables
  - the probabilities of the worlds satisfying that assignment sum to 1.



Possible worlds and decision variables

- A possible world specifies a value
  for each random variable and each decision variable
- For each assignment of values to all decision variables
  - the probabilities of the worlds satisfying that assignment sum to 1.



	Conditional probability	Utility
w0	0.2	35
w1	0.8	95
w2	0.01	
w3	0.99	
w4	0.2	
w5	0.8	
w6	0.01	
w7	0.99	

Utility

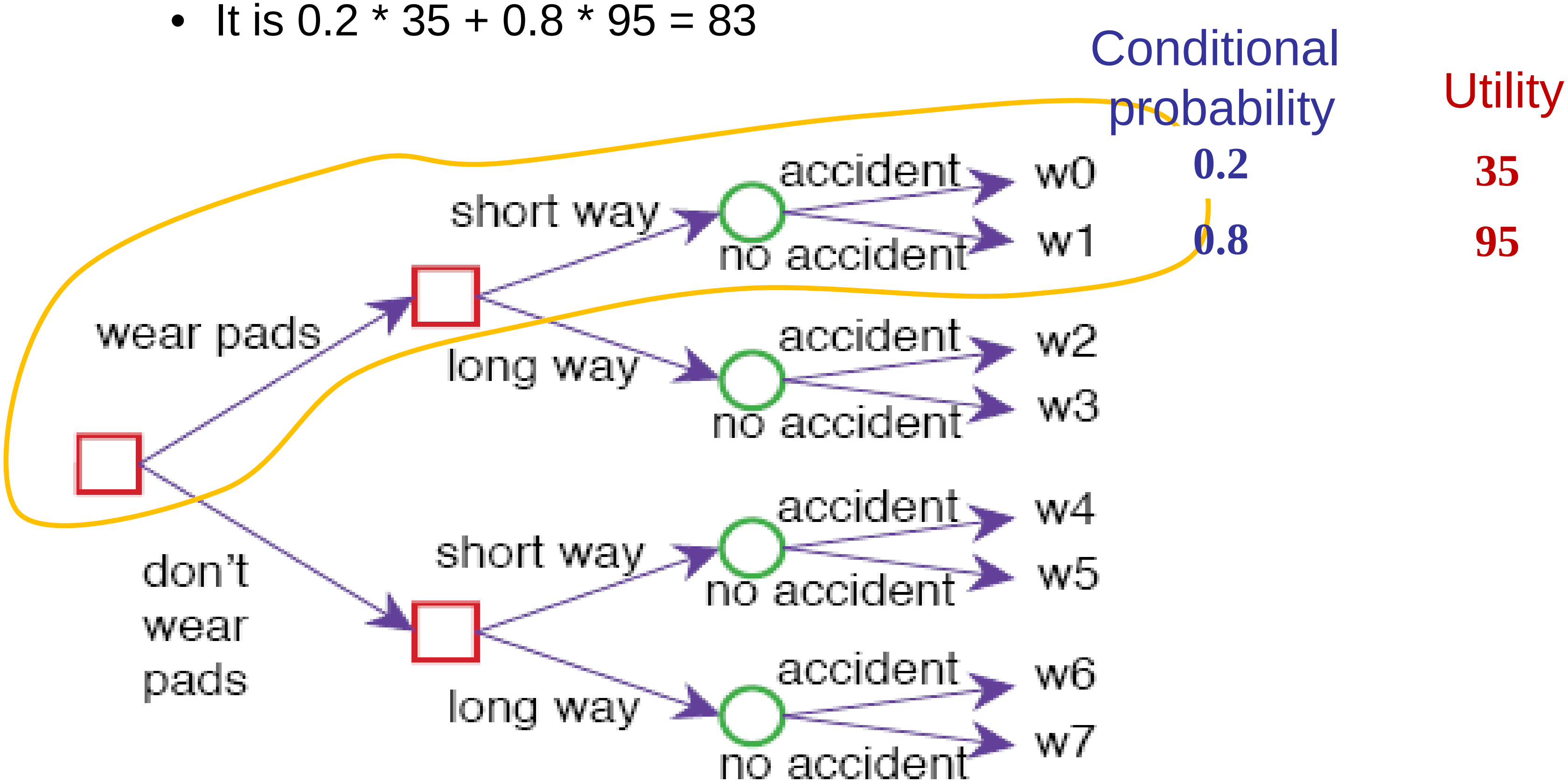
- Utility: a measure of desirability of possible worlds to an agent
  - Let U be a real-valued function such that U(w) represents an agent's degree of preference for world w
  - Expressed by a number in [0,100]

- Simple goals can still be specified
  - Worlds that satisfy the goal have utility 100
  - Other worlds have utility 0

- Utilities can be more complicated
  - For example, in the robot delivery domains, they could involve
    - Amount of damage
    - Reached the target room?
    - Energy left
    - Time taken

Combining probabilities and utilities

- We can combine probability with utility
  - The expected utility of a probability distribution over possible worlds average utility, weighted by probabilities of possible worlds
  - What is the expected utility of Wearpads=yes, Way=short ?
    - It is 0.2 * 35 + 0.8 * 95 = 83



Expected utility

- Suppose U(w) is the utility of possible world w and P(w) is the probability of possible world w

Definition (expected utility)
The expected utility is

$$E[U] = \sum_w P(w)U(w)$$

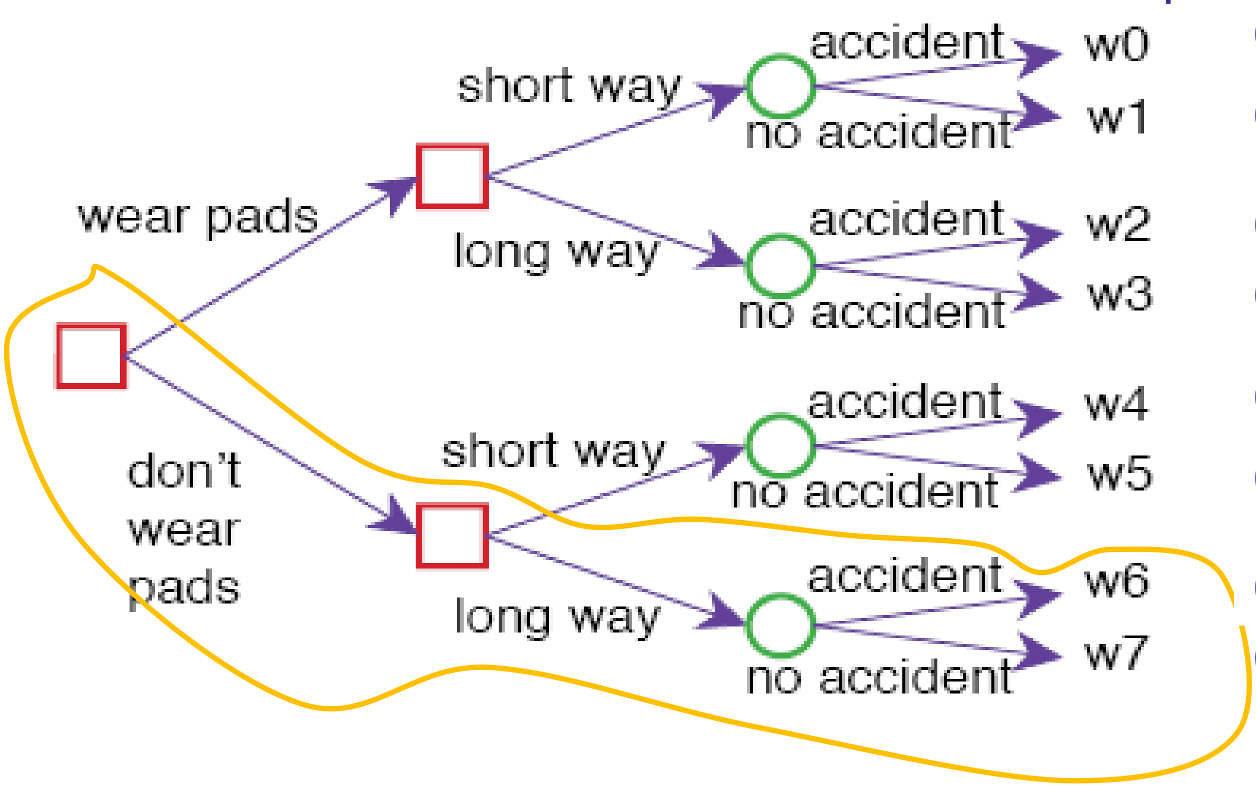
Definition (expected utility)
The conditional expected utility given e is

$$E[U|e] = \sum_w P(w|e)U(w)$$

Expected utility of a decision

- We write the expected utility of a decision as:

$$E[U|D=d] = \sum_{w} P(w|D=d)U(w)$$



	Conditional probability	Utility	E[U|D]
w0	0.2	35	83
w1	0.8	95	
w2	0.01	30	74.55
w3	0.99	75	
w4	0.2	3	80.6
w5	0.8	100	
w6	0.01	0	79.2
w7	0.99	80	

Optimal single-stage decision

- Given a single decision variable D
  - the agent can choose $D=d_i$ for any value $d_i \in dom(D)$

Definition (optimal single-stage decision)
An optimal single-stage decision is the decision $D=d_{max}$ whose expected value is maximal:

$$d_{max} \in \underset{d_i \in dom(D)}{\operatorname{argmax}} \; E[U \mid D=d_i]$$

Learning Goals For Today's Class

- Identify implied (in)dependencies in the network
- Variable elimination
  - Carry out variable elimination by using factor representation and using the factor operations
  - Use techniques to simplify variable elimination
- Define a Utility Function on possible worlds
- Define and compute optimal one-off decisions

- Assignment 4 is due on Monday
  - You should now be able to solve Questions 1, 2, 3, and 5
  - And basically Question 4
- Final exam: Monday, April 11