Branch & Bound, Summary of Advanced Topics

CPSC 322 – Search 7

Textbook § 3.7.4 and 3.7.4

January 26, 2011

Students with IDs: 99263071 & 25419094
Please come see me after the lecture
(glitch with handin)

Lecture Overview

- Some clarifications & multiple path pruning
- Recap: Iterative Deepening
- Branch & Bound

Clarifications for the A* proof

- Defined two lemmas about prefixes x of a solution path s
  - (I called the prefix pr, but a 2-letter name is confusing; let's call it x instead)

- Clarifications:
  - "Lemma":
    proven statement, stepping stone in larger proof

  - "Prefix" x of a path s:
    subpath starting from the same node as s
    - E.g. s=(a,c,z,e,d), short aczed
    - All prefixes x: a, ac, acz, acze, aczed
    - E.g. not a prefix: ab, ace, acezd (order is important!)

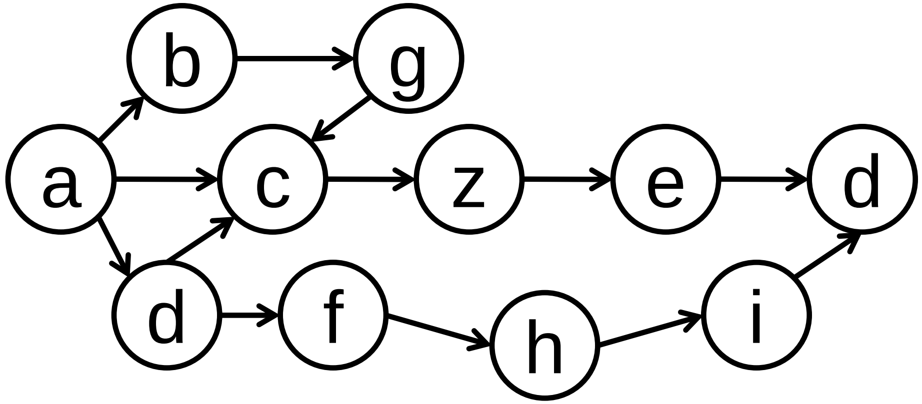
Prefixes

- Which of the following are prefixes of the path aiiscool?

aicool | ai | aii | aisc

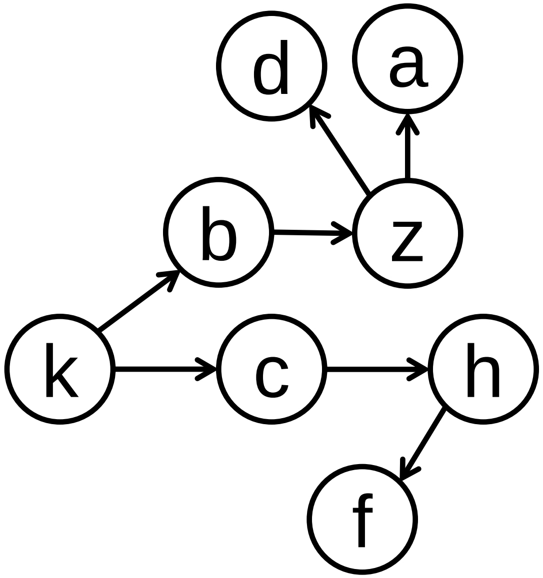
- ai and aii
- aiisc is different from aisc !
  - The optimal solution won't have a cycle if all path costs are > 0

Recap: A* admissibility

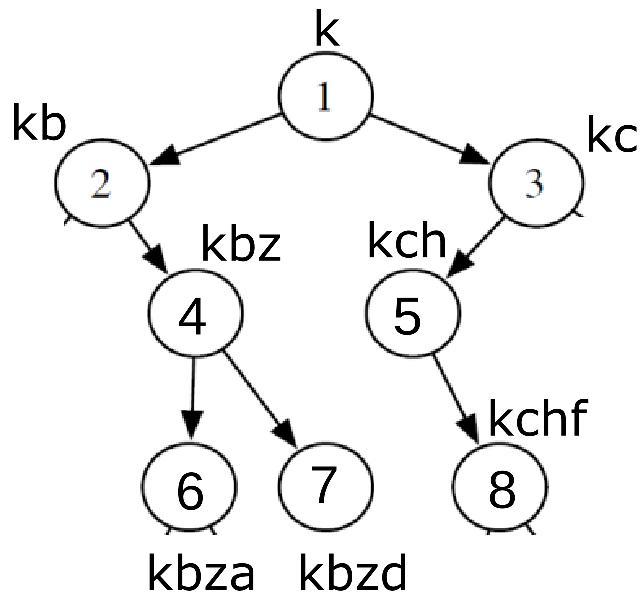


- f_{min}:= cost of optimal solution path s (e.g. s=aczed)
  - Cost is unknown but finite if a solution exists
- Lemmas for prefix x of s (exercise: prove at home)
  - Has cost $f(x) \leq f_{min}$ (due to admissibility)
  - Always one such x on the frontier (by induction)
- Used these Lemmas to prove:
  A* only expands paths x with $f(x) \leq f_{min}$
- Then we're basically done!
  - Only finite number of such paths ($\Rightarrow$ completeness)
  - Solution with cost > f_{min} won't be expanded ($\Rightarrow$ optimality)

Clarification: state space graph vs search tree



State space graph.

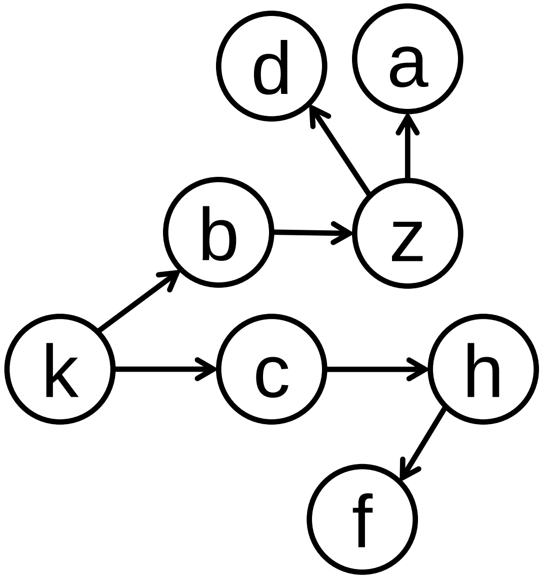


Search tree.

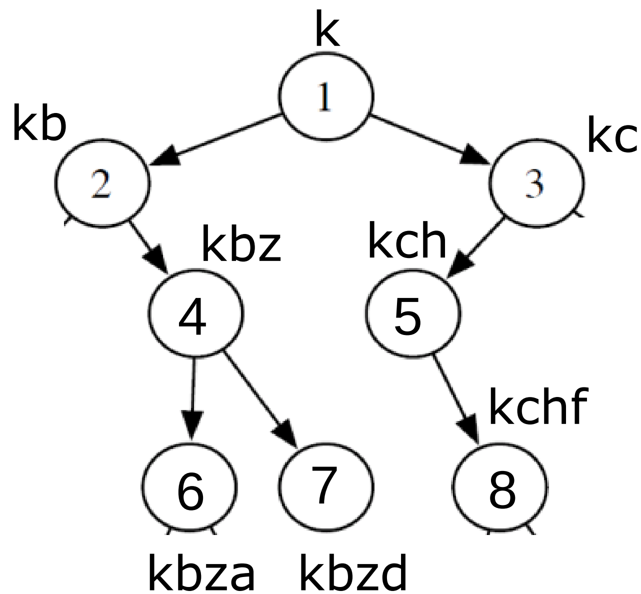
Nodes in this tree correspond to paths in the state space graph

If there are no cycles, the two look the same

Clarification: state space graph vs search tree



State space graph.



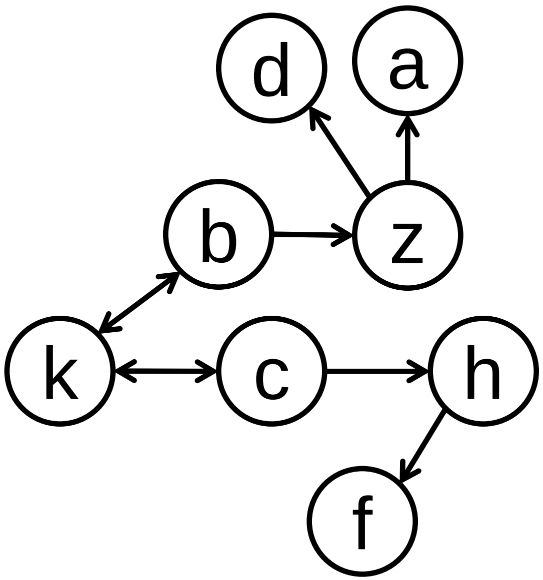
Search tree.

What do I mean by the numbers in the search tree's nodes?

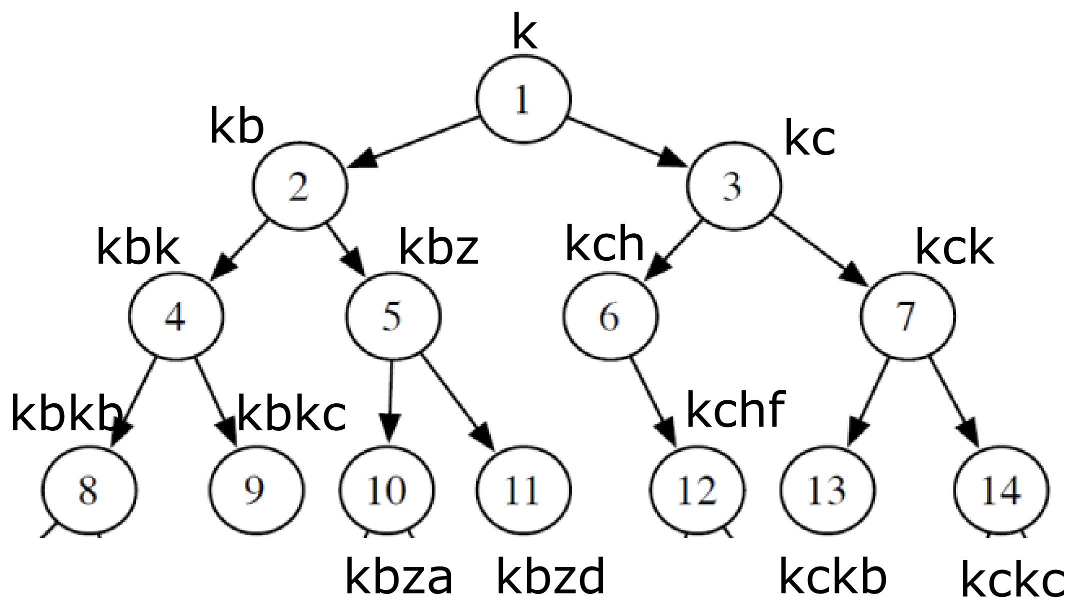
Node's name

Order in which a search algo. (here: BFS) expands nodes

Clarification: state space graph vs search tree



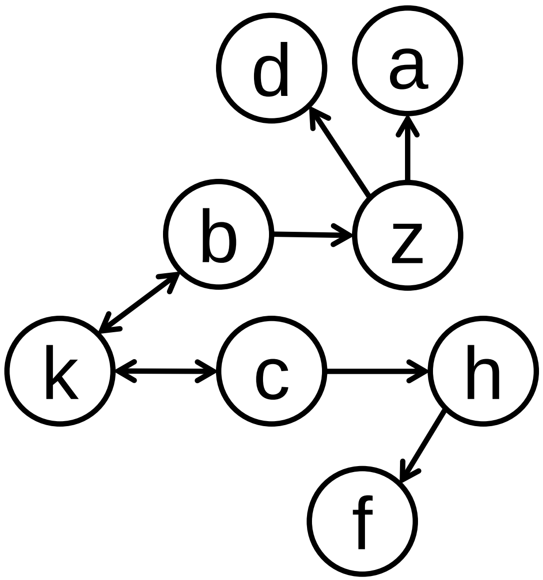
State space graph.



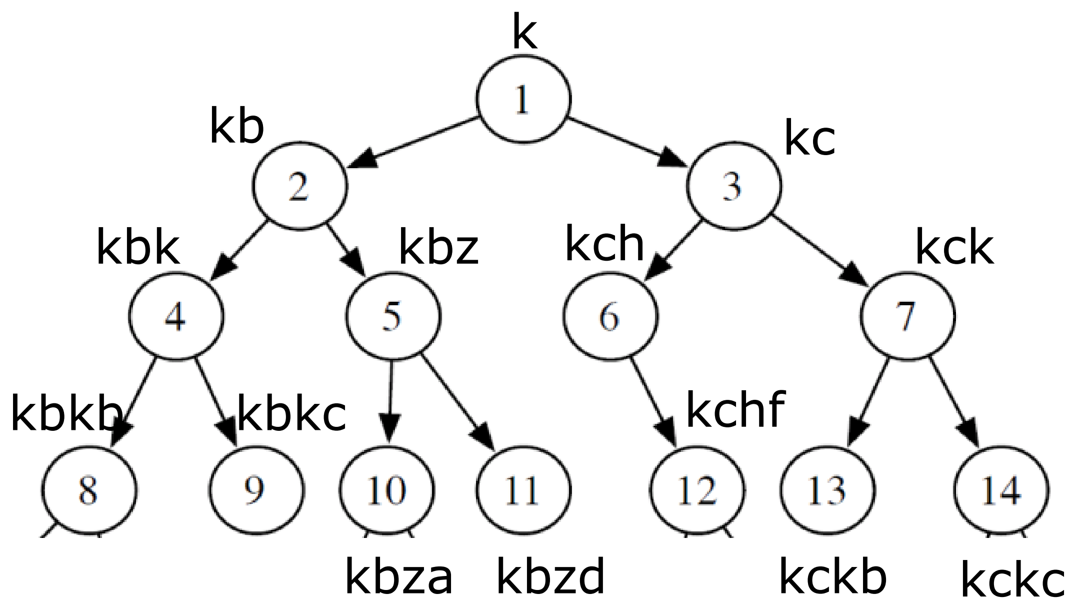
Search tree.
(only first 3 levels, of BFS)

- If there are cycles, the two look very different

Clarification: state space graph vs search tree



State space graph.

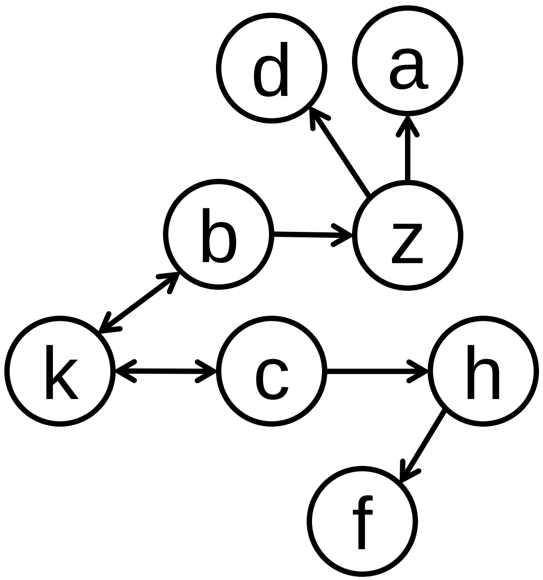


Search tree.
(only first 3 levels, of BFS)

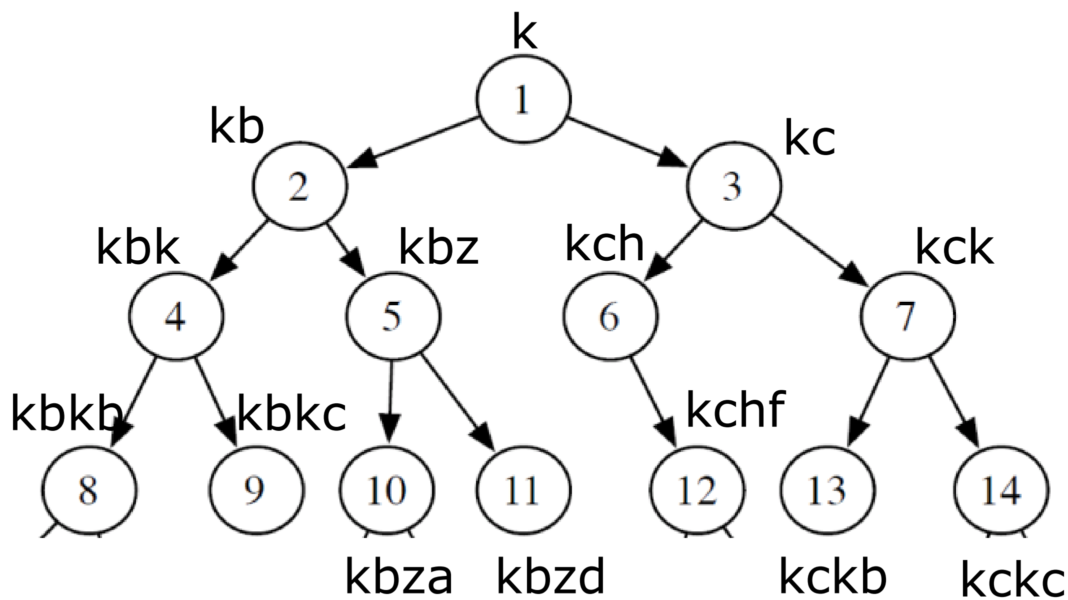
What do nodes in the search tree represent in the state space?

nodes edges paths states

Clarification: state space graph vs search tree



State space graph.

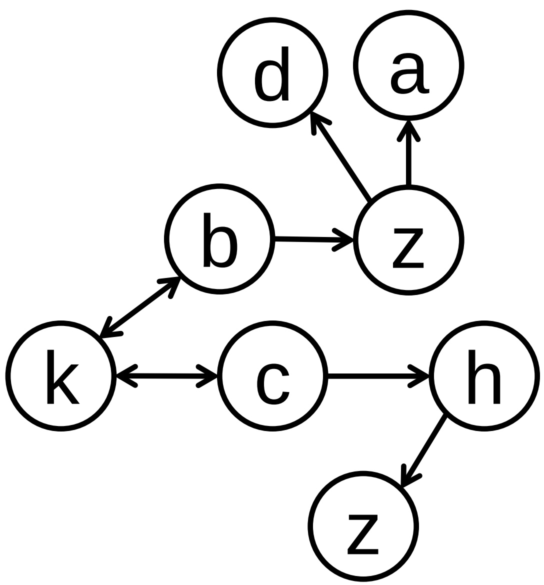


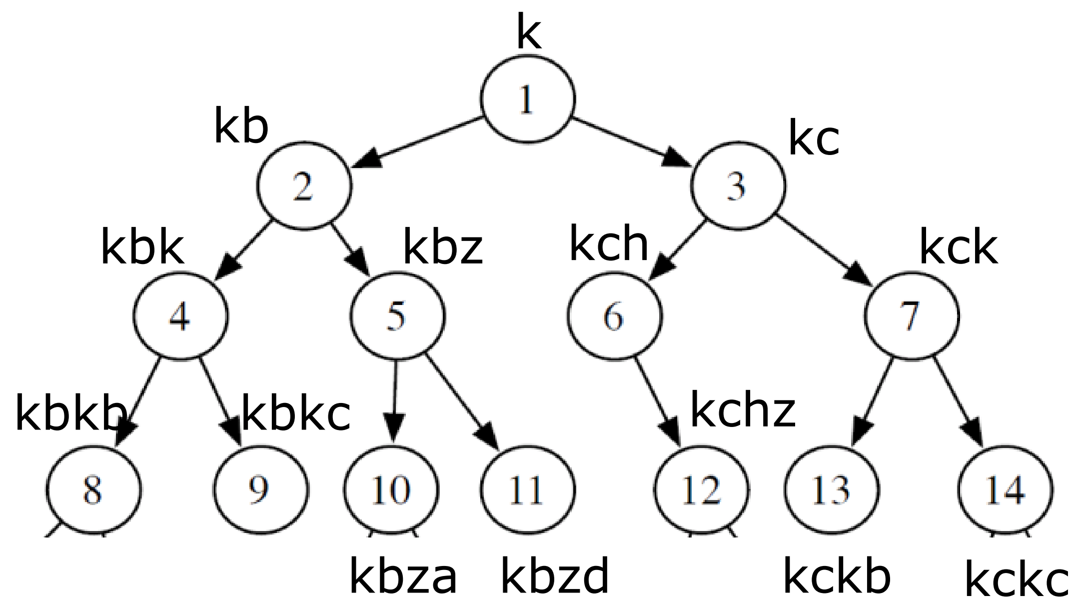
Search tree.
(only first 3 levels, of BFS)

What do edges in the search tree represent in the state space?

nodes | edges | paths | states

Clarification: state space graph vs search tree





State space graph.

Search tree.

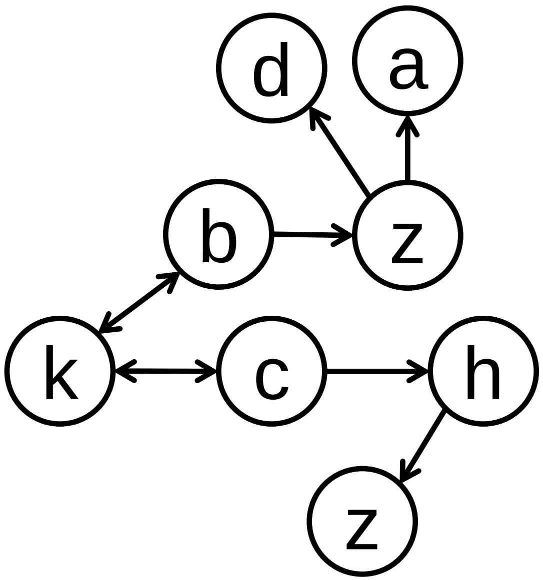
Nodes in this tree correspond to paths in the state space graph

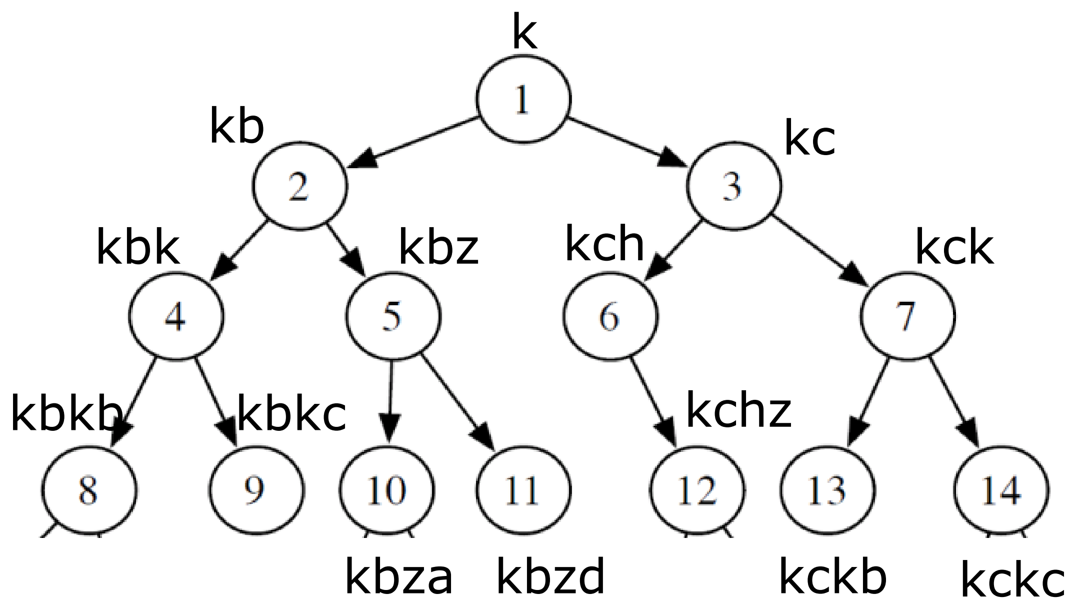
(if multiple start nodes: forest)

May contain cycles!

Cannot contain cycles!

Clarification: state space graph vs search tree





State space graph.

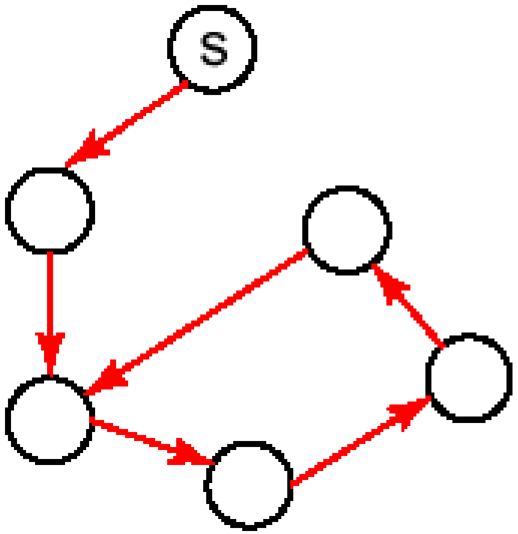
Search tree.

Nodes in this tree correspond to paths in the state space graph

Why don't we just eliminate cycles?

Sometimes (but not always) we want multiple solution paths

Cycle Checking: if we only want optimal solutions



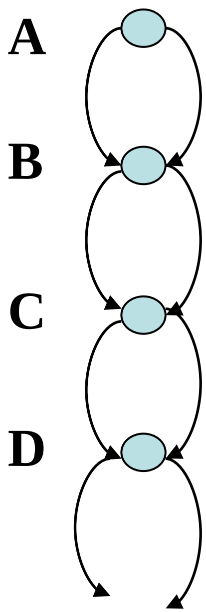
- You can prune a node *n* that is on the path from the start node to n.
- This pruning cannot remove an optimal solution ⇒ cycle check

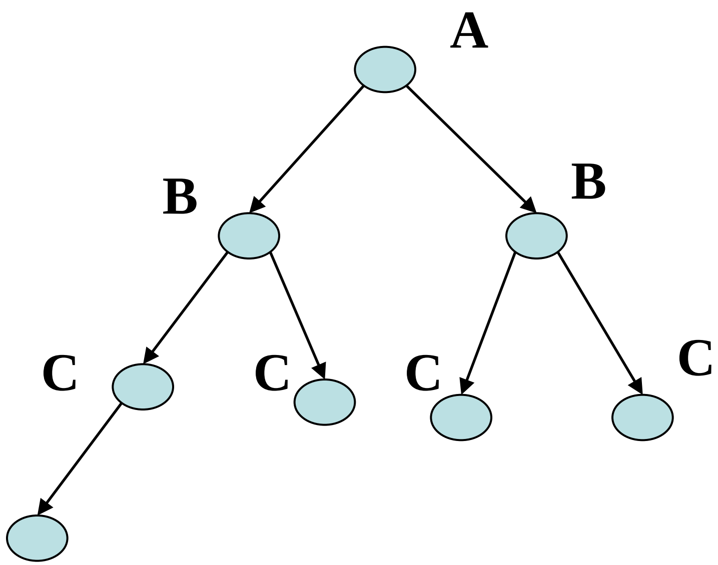
- Using depth-first methods, with the graph explicitly stored, this can be done in constant time
  - Only one path being explored at a time

- Other methods: cost is linear in path length
  - (check each node in the path)

Size of search space vs search tree

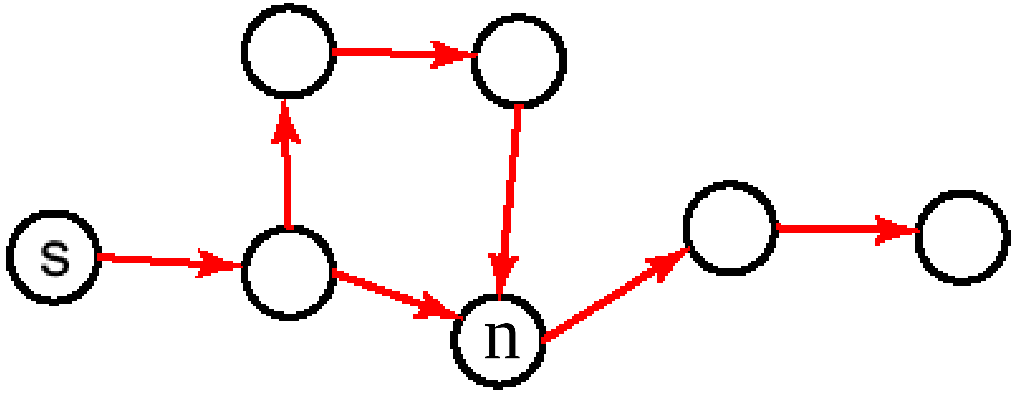
- With cycles, search tree can be exponential in the state space
  - E.g. state space with 2 actions from each state to next
  - With d + 1 states, search tree has depth d





- **2^d possible paths through the search space => exponentially larger search tree!**

Multiple Path Pruning

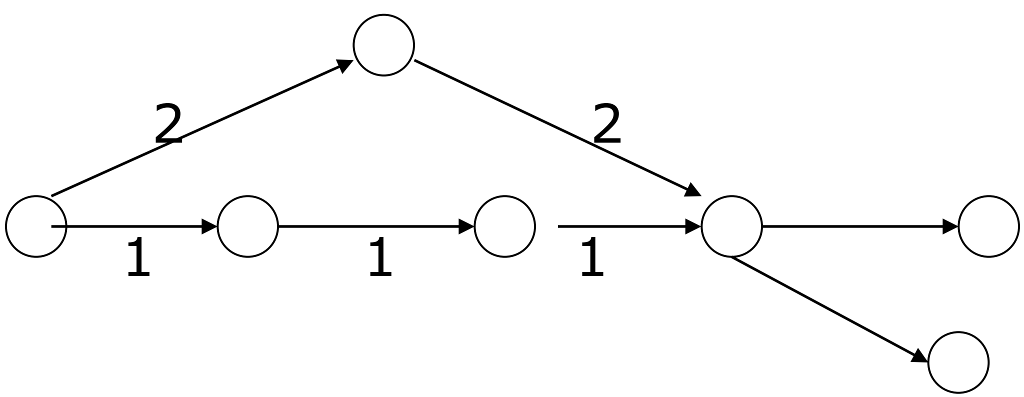


- If we only want one path to the solution
- Can prune path to a node *n* that has already been reached via a previous path
  - Store S := {all nodes n that have been expanded}
  - For newly expanded path p = $(n_1,\ldots,n_k,n)$
    - Check whether n $\in$ S
    - Subsumes cycle check
- Can implement by storing the path to each expanded node

Multiple-Path Pruning & Optimal Solutions

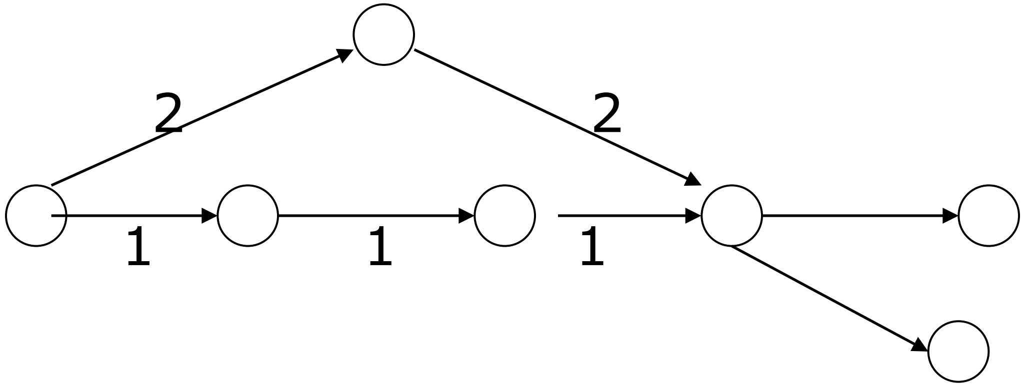
- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want an optimal solution ?

- Can remove all paths from the frontier that use the longer path. (these can't be optimal)



Multiple-Path Pruning & Optimal Solutions

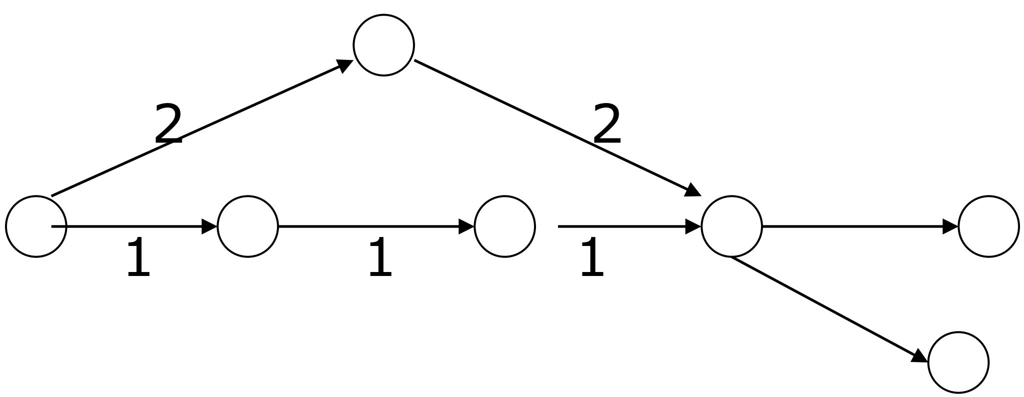
- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want just the optimal solution ?
- Can change the initial segment of the paths on the frontier to use the shorter path



Multiple-Path Pruning & Optimal Solutions

- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want just the optimal solution ?

- Can prove that this can't happen for an algorithm



- Which of the following algorithms always find the shortest path to nodes on the frontier first?

Least Cost Search First

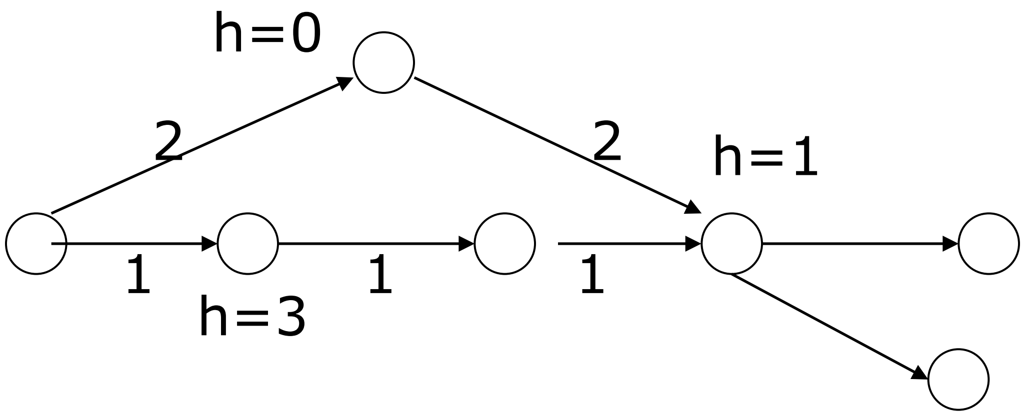
A*

Both of the above

None of the above

- Which of the following algorithms always find the shortest path to nodes on the frontier first?
  - Only Least Cost First Search (like Dijkstra’s algorithm)
  - For A* this is only guaranteed for nodes on the optimal solution path

  - Example: A* expands the upper path first
    - Special conditions on the heuristic can recover the guarantee of LCFS



Summary: pruning

- Sometimes we don’t want pruning
  - Actually want multiple solutions (including non-optimal ones)
- Search tree can be exponentially larger than search space
  - So pruning is often important
- In DFS-type search algorithms
  - We can do cheap cycle checks: O(1)
- BFS-type search algorithms are memory-heavy already
  - We can store the path to each expanded node and do multiple path pruning

Lecture Overview

- Some clarifications & multiple path pruning
- Recap: Iterative Deepening
- Branch & Bound

Iterative Deepening DFS (short IDS): Motivation

Want low space complexity but completeness and optimality

Key Idea: re-compute elements of the frontier
rather than saving them

	Complete	Optimal	Time	Space
DFS	N (Y if no cycles)	N	$O(b^m)$	$O(mb)$
BFS	Y	Y	$O(b^m)$	$O(b^m)$
LCFS (when arc costs available)	Y Costs > 0	Y Costs >=0	$O(b^m)$	$O(b^m)$
Best First (when *h* available)	N	N	$O(b^m)$	$O(b^m)$
A* (when arc costs and *h* available)	Y Costs > 0 *h* admissible	Y Costs >=0 *h* admissible	$O(b^m)$	$O(b^m)$

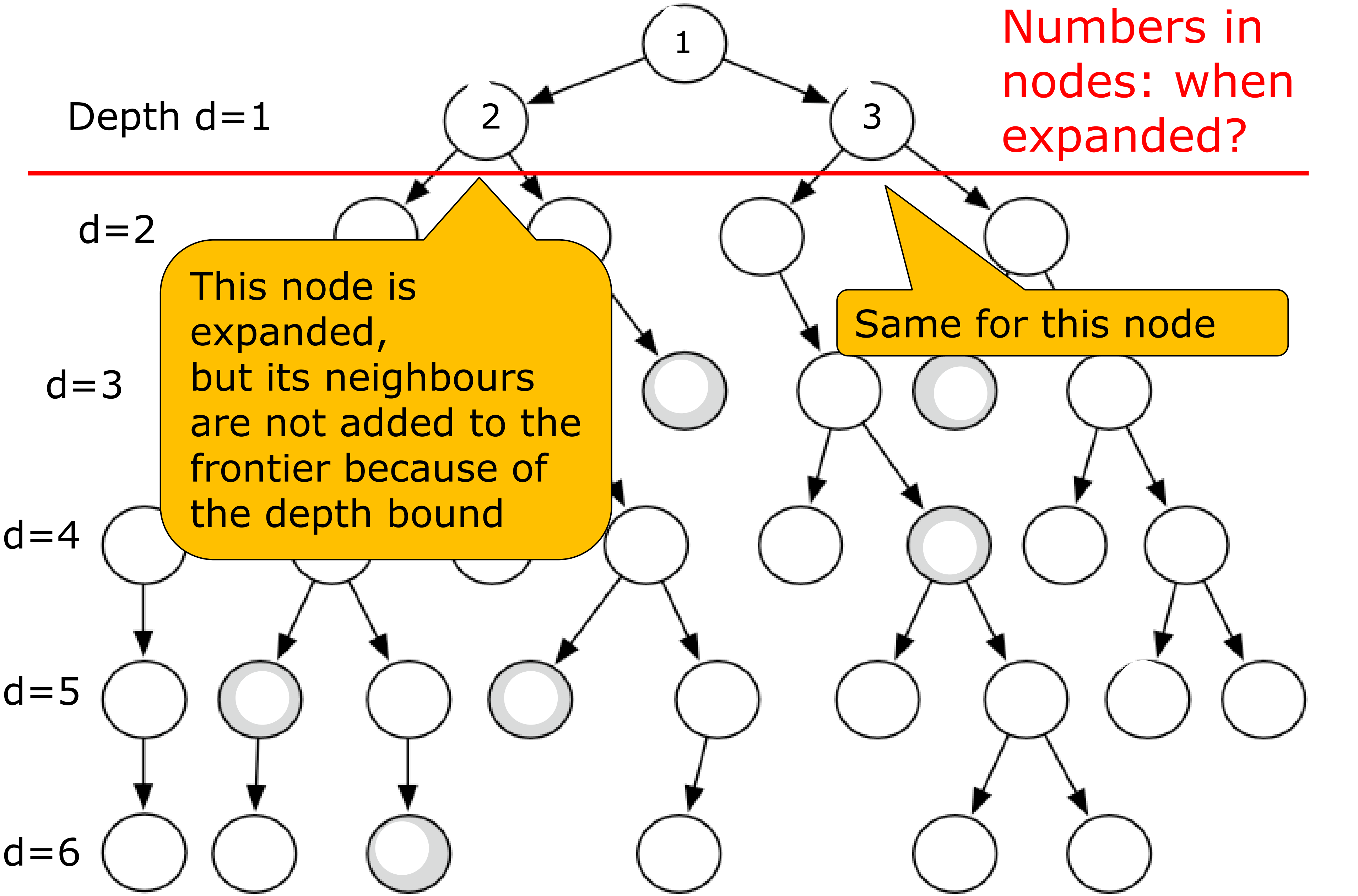
Iterative Deepening DFS (IDS) in a Nutshell

- Depth-bounded depth-first search: DFS on a leash
  - For depth bound d, ignore any paths with longer length:
    - Not allowed to go too far away ⇒ backtrack ("fail unnaturally")
    - Only finite # paths with length ≤ d ⇒ terminates
  - What is the memory requirement at depth bound d? (it is DFS!)
    - m=length of optimal solution path
    - b=branching factor

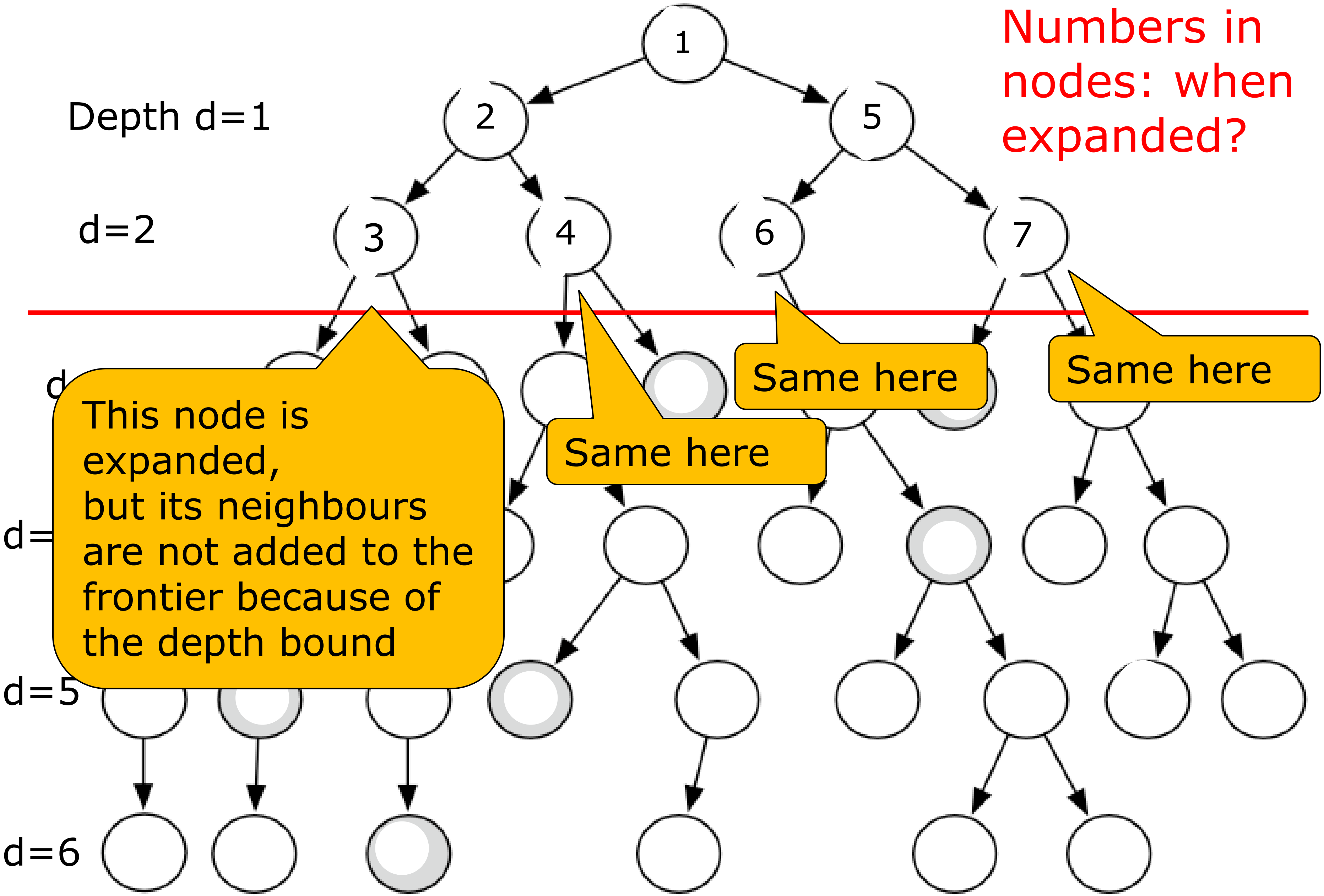
$O(b^m)$	$O(mb)$	$O(bd)$	$O(dm)$

    - O(bd) ! It's a DFS, up to depth d.
- Progressively increase the depth bound d
  - Start at 1
  - Then 2
  - Then 3
  - ...
  - Until it finds the solution at depth m

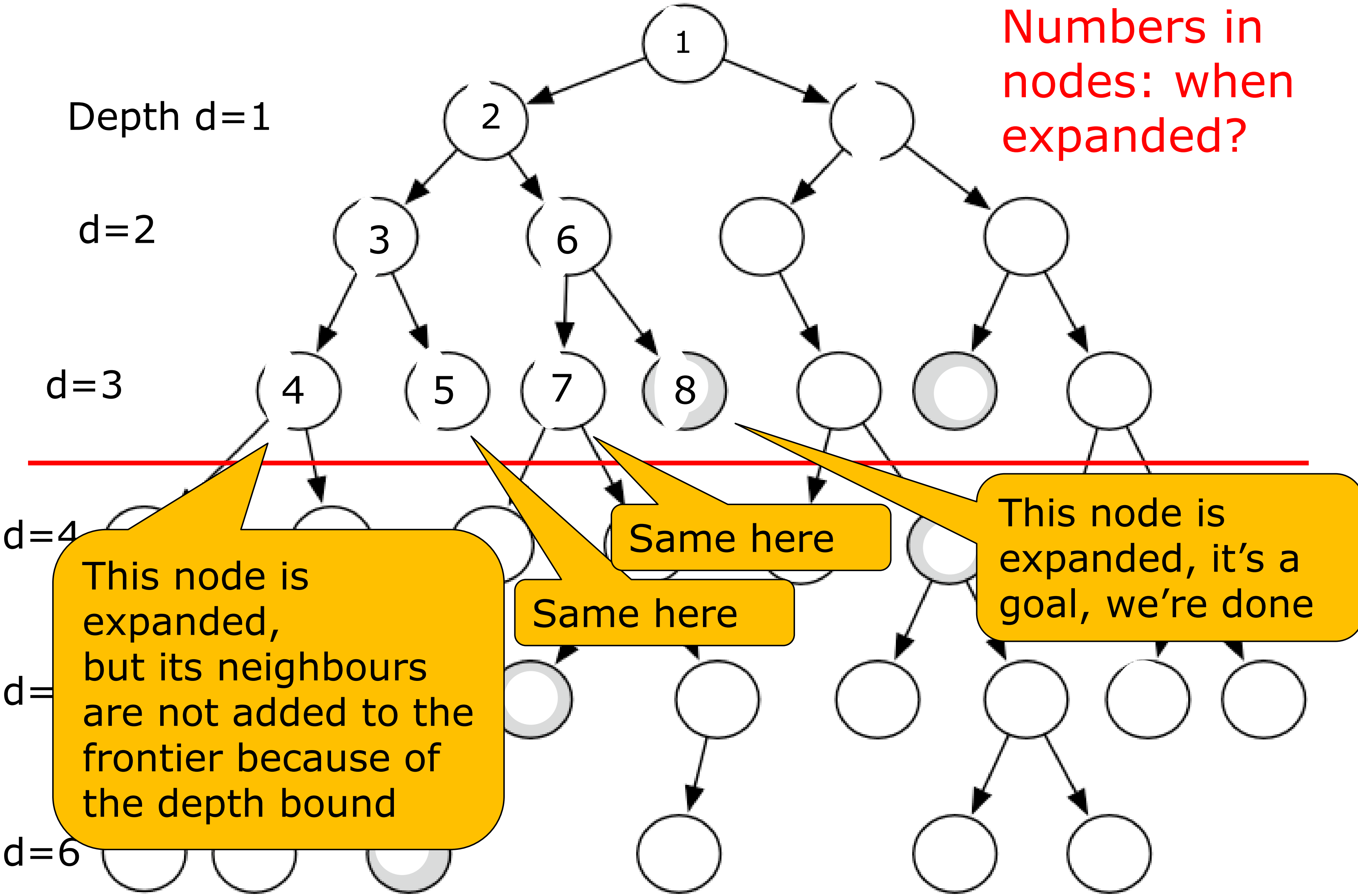
Iterative Deepening DFS, depth bound = 1



Iterative Deepening DFS, depth bound = 2



Iterative Deepening DFS, depth bound = 3



Analysis of Iterative Deepening DFS (IDS)

- Space complexity

  $O(b^m)$ $O(m^b)$ $O(bm)$ $O(b+m)$

  – DFS scheme, only explore one branch at a time

- Complete? Yes No

  – Only finite # of paths up to depth m, doesn't explore longer paths

- Optimal? Yes No

  – Proof by contradiction

(Time) Complexity of IDS

The solution is at depth m, branching factor b

Total # of paths generated:

$$\leq b^{m} + (2\ b^{m-1}) + (3\ b^{m-2}) + ... + mb$$

We only expand paths at depth m once

We only expand paths at depth m-1 twice

We only expand paths at depth m-2 three times

We expand paths at depth 1 m times (for every single depth bound)

(Time) Complexity of IDS

From there on, it's just math:

Total # paths generated by IDS

$\leq\ b^m + (2\ b^{m-1}) + (3\ b^{m-2}) + \ldots + mb$

$= b^m\ (1\ b^0 + 2\ b^{-1} + 3\ b^{-2} + \ldots + m\ b^{1-m}\ )$

$$= b^m\left(\sum_{i=1}^{m} i b^{1-i}\right) = b^m\left(\sum_{i=1}^{m} i (b^{-1})^{i-1}\right)$$

$$\leq b^m\left(\sum_{i=0}^{\infty} i (b^{-1})^{i-1}\right) = b^m\left(\frac{1}{1-b^{-1}}\right)^2 = b^m\left(\frac{b}{b-1}\right)^2 \in O(b^m)$$

If b > 1

Geometric progression: for |r|<1: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$

$$\frac{d}{dr}\sum_{i=0}^{\infty} r^i = \sum_{i=0}^{\infty} i r^{i-1} = \frac{1}{(1-r)^2}$$

Conclusion for Iterative Deepening

- Even though it redoes what seems like a lot of work
  - Actually, compared to how much work there is at greater depths, it's not a lot of work
  - Redoes the first levels most often
    - But those are the cheapest ones

- Time Complexity $O(b^m)$
  - Just like a single DFS
  - Just like the last depth-bounded DFS
    - That last depth bounded DFS dominates the search complexity

- Space complexity: $O(bm)$
- Optimal
- Complete

(Heuristic) Iterative Deepening: IDA*

- Like Iterative Deepening DFS
  - But the "depth" bound is measured in terms of the f value
  - f-value-bounded DFS: DFS on a f-value leash
  - IDA* is a bit of a misnomer
    - The only thing it has in common with A* is that it uses the f value $f(p) = cost(p) + h(p)$
    - It does NOT expand the path with lowest f value. It is doing DFS!
    - But f-value-bounded DFS doesn't sound as good …

- If you don't find a solution at a given f-value
  - Increase the bound:
    to the minimum of the f-values that exceeded the previous bound

- Will explore all nodes with f value $< f_{min}$ (optimal one)

Analysis of Iterative Deepening A* (IDA*)

- Complete and optimal? Same conditions as A*
  - h is admissible
  - all arc costs > 0
  - finite branching factor

- Time complexity: $O(b^m)$
  - Same argument as for Iterative Deepening DFS

- Space complexity:

  $O(b^m)$ $O(m^b)$ $O(bm)$ $O(b+m)$

  - Same argument as for Iterative Deepening DFS

Search methods so far

	Complete	Optimal	Time	Space
DFS	N (Y if no cycles)	N	$O(b^m)$	$O(mb)$
BFS	Y	Y	$O(b^m)$	$O(b^m)$
IDS	Y	Y	$O(b^m)$	$O(mb)$
LCFS (when arc costs available)	Y Costs > 0	Y Costs >=0	$O(b^m)$	$O(b^m)$
Best First (when *h* available)	N	N	$O(b^m)$	$O(b^m)$
A* (when arc costs and *h* available)	Y Costs > 0 *h* admissible	Y Costs >=0 *h* admissible	$O(b^m)$	$O(b^m)$
IDA*	Y (same cond. as A*)	Y	$O(b^m)$	$O(mb)$

Lecture Overview

- Some clarifications & multiple path pruning
- Recap: Iterative Deepening
- Branch & Bound

Heuristic DFS

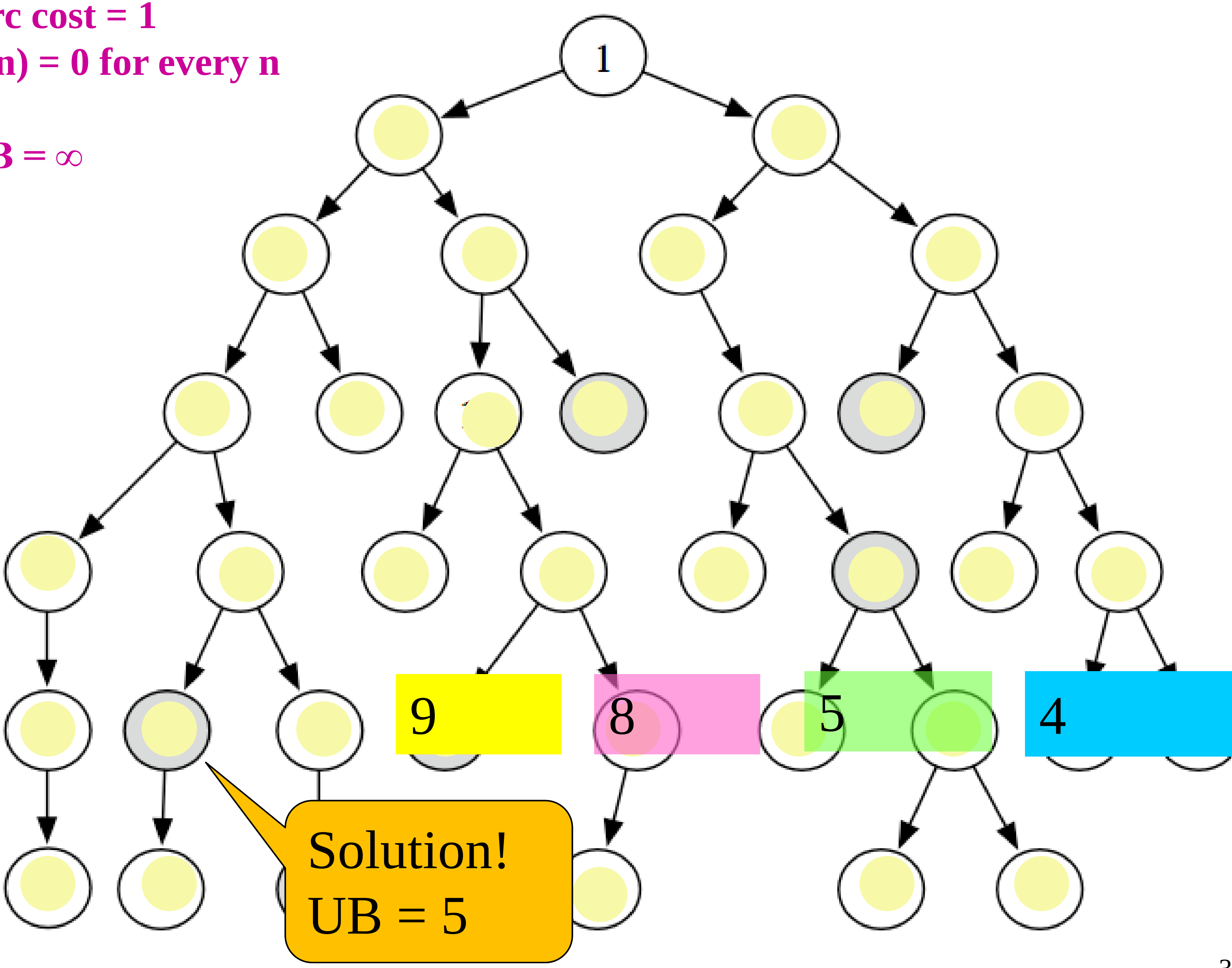
- Other than IDA*, can we use heuristic information in DFS?
  - When we expand a node, we put all its neighbours on the stack
  - In which order?
    - Can use heuristic guidance: h or f
    - Perfect heuristic: would solve problem without any backtracking

- Heuristic DFS is very frequently used in practice
  - Often don’t need optimal solution, just some solution
  - No requirement for admissibility of heuristic

Branch-and-Bound Search

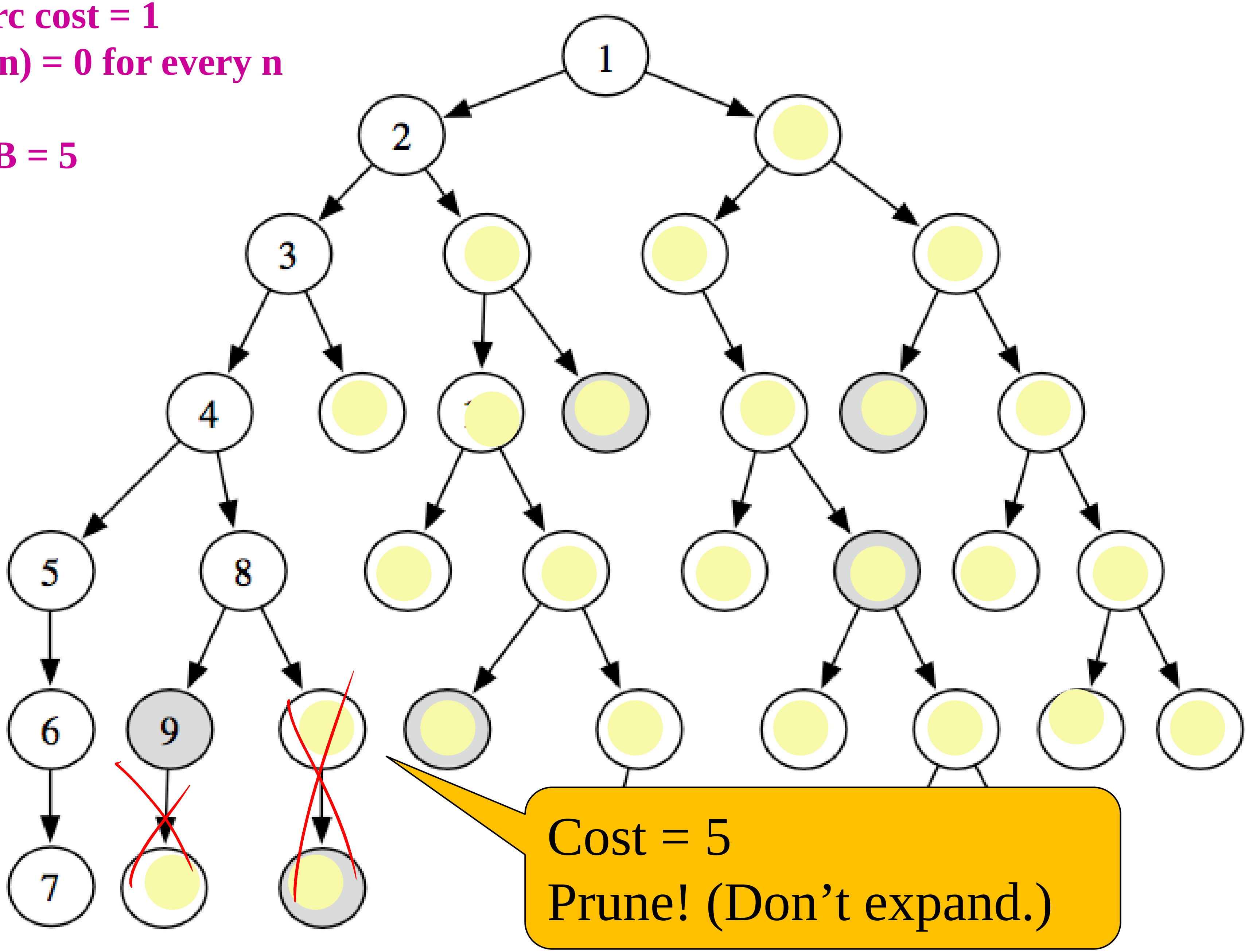
- Another way to combine DFS with heuristic guidance
- Follows exactly the same search path as depth-first search
  - But to ensure optimality, it does not stop at the first solution found
- It continues, after recording upper bound on solution cost
  - upper bound: *UB* = cost of the best solution found so far
  - Initialized to ∞ or any overestimate of solution cost
- When a path *p* is selected for expansion:
  - Compute LB(p) = f(p) = cost(p) + h(p)
    - If LB(p) $\geq$UB, remove *p* from frontier without expanding it (pruning)
    - Else expand *p*, adding all of its neighbors to the frontier
  - Requires admissible h

- Arc cost = 1
- h(n) = 0 for every n
- UB = ∞

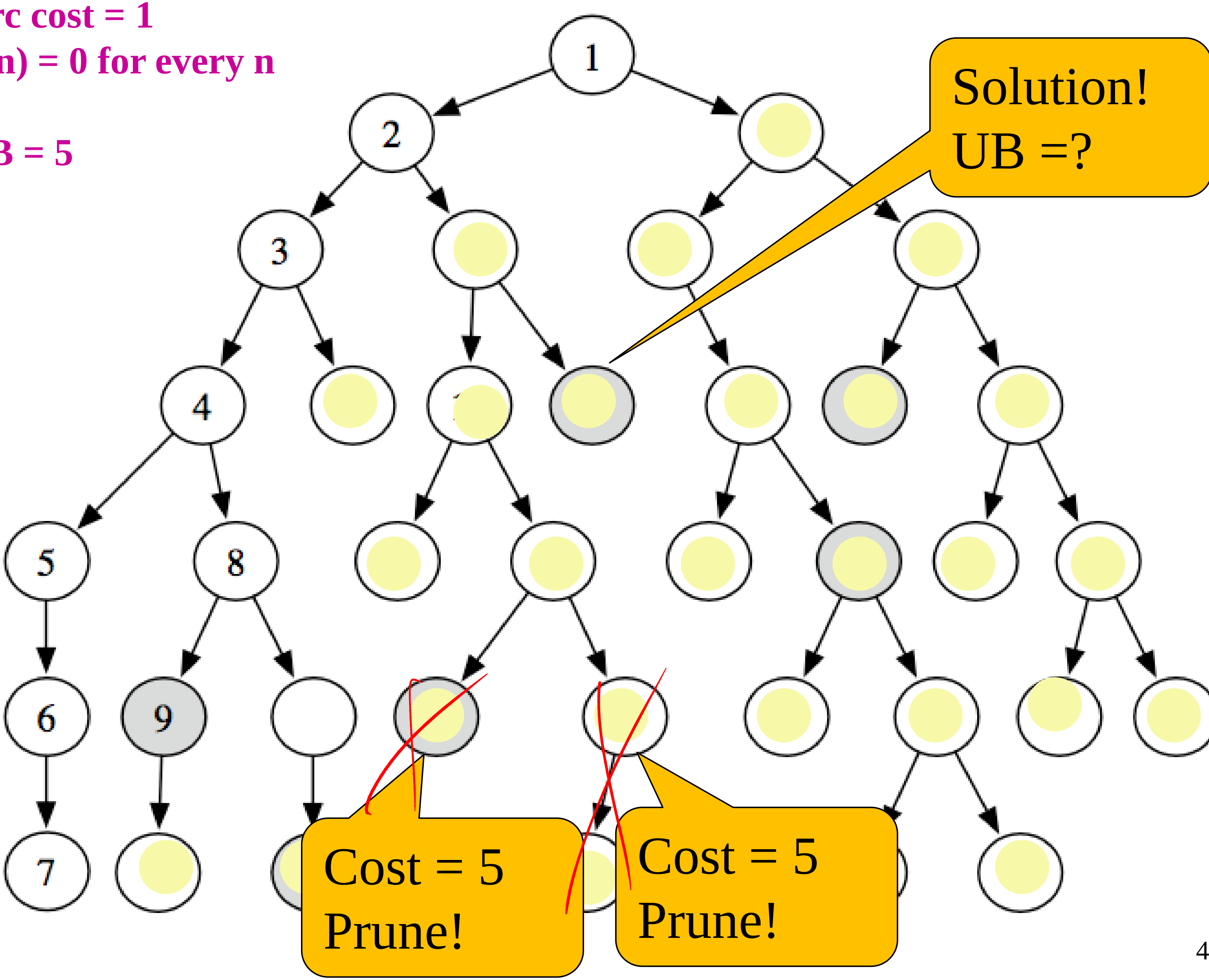


- Arc cost = 1
- h(n) = 0 for every n

- UB = 5

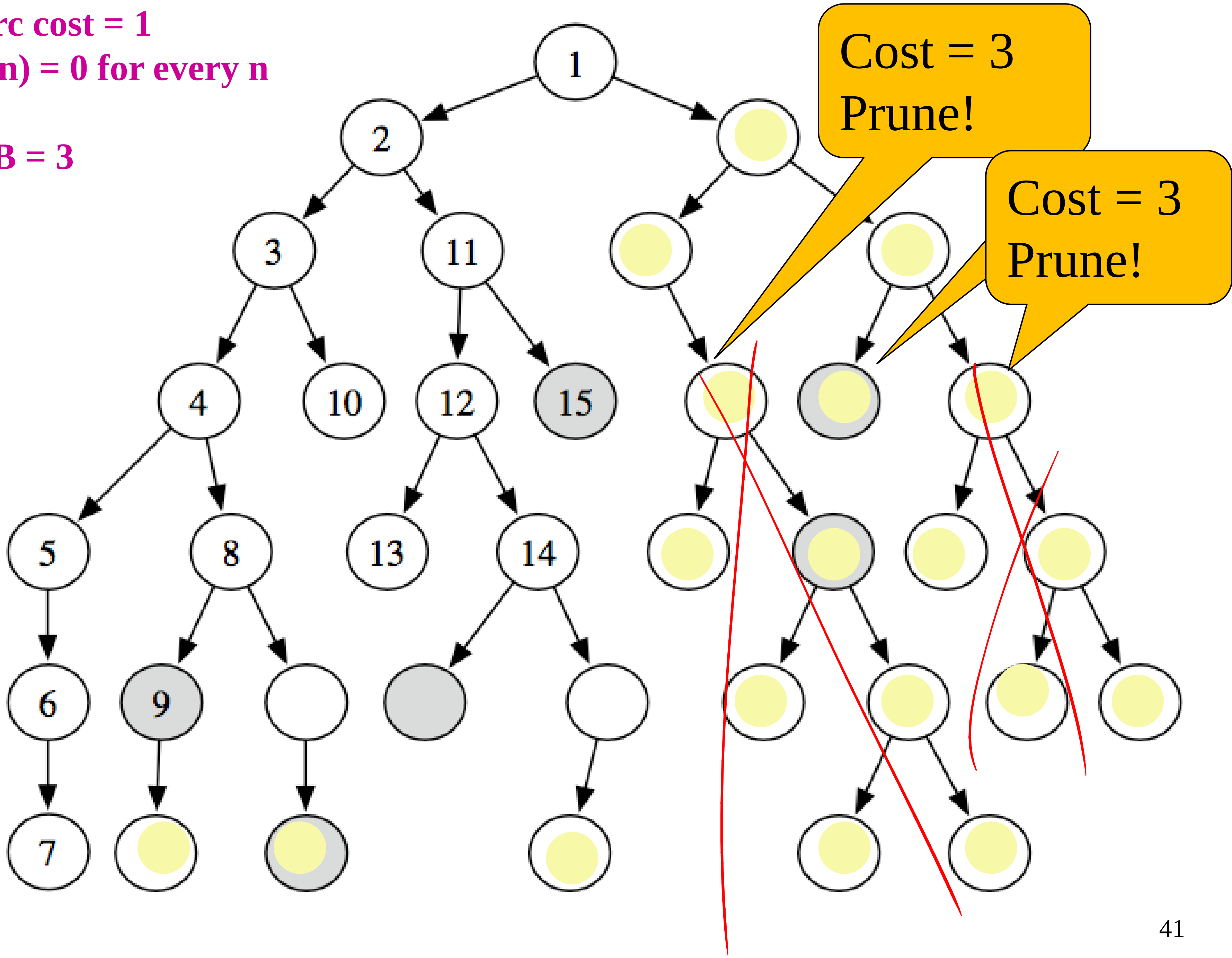


- Arc cost = 1
- h(n) = 0 for every n
- UB = 5



- **Arc cost = 1**
- **h(n) = 0 for every n**

- **UB = 3**



Branch-and-Bound Analysis

- Complete? **YES** **NO** **IT DEPENDS**
  - Same as DFS: can't handle infinite graphs.
  - But complete if initialized with some finite UB
- Optimal? **YES** **NO** **IT DEPENDS**
  - YES.
- Time complexity: $O(b^m)$
- Space complexity
  - It's a DFS

$O(b^m)$ $O(m^b)$ $O(bm)$ $O(b+m)$

Combining B&B with heuristic guidance

- We said
  - “Follows exactly the same search path as depth-first search”“
  - Let’s make that heuristic depth-first search
- Can freely choose order to put neighbours on the stack
  - Could e.g. use a separate heuristic h’ that is NOT admissible
- To compute LB(p)
  - Need to compute f value using an admissible heuristic h
- This combination is used a lot in practice
  - Sudoku solver in assignment 2 will be along those lines
  - But also integrates some logical reasoning at each node

Search methods so far

	Complete	Optimal	Time	Space
DFS	N (Y if no cycles)	N	$O(b^m)$	$O(mb)$
BFS	Y	Y	$O(b^m)$	$O(b^m)$
IDS	Y	Y	$O(b^m)$	$O(mb)$
LCFS (when arc costs available)	Y Costs > 0	Y Costs >=0	$O(b^m)$	$O(b^m)$
Best First (when *h* available)	N	N	$O(b^m)$	$O(b^m)$
A* (when arc costs and *h* available)	Y Costs > 0 *h* admissible	Y Costs >=0 *h* admissible	$O(b^m)$	$O(b^m)$
IDA*	Y (same cond. as A*)	Y	$O(b^m)$	$O(mb)$
Branch & Bound	N (Y if init. with finite UB)	Y	$O(b^m)$	$O(mb)$

Learning Goals for today's class

- Define/read/write/trace/debug different search algorithms
  - In more detail today: Iterative Deepening,
    New today: Iterative Deepening A*, Branch & Bound
- Apply basic properties of search algorithms:
  - completeness, optimality, time and space complexity

Announcements:

- New practice exercises are out: see WebCT
  - Heuristic search
  - Branch & Bound
  - Please use these! (Only takes 5 min. if you understood things…)
- Assignment 1 is out: see WebCT

Learning Goals for search

- **Identify** real world examples that make use of deterministic, goal-driven search agents
- **Assess** the size of the search space of a given search problem.
- **Implement** the generic solution to a search problem.
- **Apply** basic properties of search algorithms:
  - completeness, optimality, time and space complexity
- **Select** the most appropriate search algorithms for specific problems.
- **Define/read/write/trace/debug** different search algorithms
- **Construct** heuristic functions for specific search problems
- **Formally prove** A* optimality.
- **Define optimally** efficient

Learning goals: know how to fill this

	Selection	Complete	Optimal	Time	Space
DFS					
BFS					
IDS					
LCFS					
Best First					
A*					
B&B					
IDA*					

Memory-bounded *A**

- Iterative deepening A* and B & B use little memory
- What if we've got more memory, but not $O(b^m)$?
- Do A* and keep as much of the frontier in memory as possible
- When running out of memory
  - delete worst path (highest f value) from frontier
  - Back its *f* value up to a common ancestor
- Subtree gets regenerated only when all other paths have been shown to be worse than the “forgotten” path

- Details are beyond the scope of the course, but
  - Complete and optimal if solution is at depth manageable for available memory

Algorithms Often Used in Practice

	Selection	Complete	Optimal	Time	Space
DFS	LIFO	N	N	$O(b^m)$	$O(mb)$
BFS	FIFO	Y	Y	$O(b^m)$	$O(b^m)$
IDS	LIFO	Y	Y	$O(b^m)$	$O(mb)$
LCFS	min cost	Y **	Y **	$O(b^m)$	$O(b^m)$
Best First	min h	N	N	$O(b^m)$	$O(b^m)$
A*	min f	Y**	Y**	$O(b^m)$	$O(b^m)$
B&B	LIFO + pruning	N (Y if UB finite)	Y	$O(b^m)$	$O(mb)$
IDA*	LIFO	Y	Y	$O(b^m)$	$O(mb)$
MBA*	min f	Y**	Y**	$O(b^m)$	$O(b^m)$

** Needs conditions

Coming up: Constraint Satisfaction Problems

- Read chapter 4
- Student with IDs: 99263071 & 25419094
  - Please come see me (glitch with handin)